Solutions to the Homework in Lecture 8
Fundamentals of Observational Cosmology (II)

These notes provide worked solutions to the homework problems in Lecture 8. We use the standard definitions from the lecture notes,

$$\mu \equiv \frac{k_\parallel}{k}, \qquad \beta \equiv \frac{f}{b}, \qquad P_g^s(k,\mu) = b^2(1+\beta\mu^2)^2 P_m(k).$$

Problem 1. Real to redshift space

Question. Starting from

$$\mathbf{s} = \mathbf{r} + \frac{v_\parallel}{aH}\hat{\mathbf{n}},$$

explain why only the line-of-sight coordinate is distorted.

Solution. The observed redshift tells us primarily about a galaxy's motion *along the line of sight*. In the mapping

$$\mathbf{s} = \mathbf{r} + \frac{v_\parallel}{aH}\hat{\mathbf{n}},$$

the correction term is proportional to $\hat{\mathbf{n}}$, the unit vector pointing from the observer to the galaxy. Therefore the peculiar velocity only shifts the position in the radial direction.

To make this explicit, decompose the position into transverse and radial parts:

$$\mathbf{r} = \mathbf{r}_\perp + r_\parallel \hat{\mathbf{n}}.$$

Then

$$\mathbf{s} = \mathbf{r}_\perp + \left(r_\parallel + \frac{v_\parallel}{aH}\right)\hat{\mathbf{n}}.$$

Hence

$$\boxed{\mathbf{s}_\perp = \mathbf{r}_\perp, \qquad s_\parallel = r_\parallel + \frac{v_\parallel}{aH}.}$$

So the transverse coordinates are unchanged, while only the line-of-sight coordinate is shifted. Physically, this happens because peculiar velocities alter the observed redshift, and the redshift is used to infer radial distance. A velocity perpendicular to the line of sight does not contribute to the Doppler redshift at this order.

Problem 2. Kaiser derivation

Question. Starting from

$$\delta_g^s(\mathbf{k}) = b(1+\beta\mu^2)\delta_m(\mathbf{k}),$$

derive

$$P_g^s(k,\mu) = b^2(1+\beta\mu^2)^2 P_m(k).$$

Solution. The power spectrum is defined by the two-point function in Fourier space:

$$\langle \delta(\mathbf{k})\delta^*(\mathbf{k}')\rangle = (2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')P(k).$$

Apply this to the redshift-space galaxy field. Since

$$\delta_g^s(\mathbf{k}) = b(1+\beta\mu^2)\delta_m(\mathbf{k}),$$

we have

$$\delta_g^{s*}(\mathbf{k}') = b(1+\beta\mu'^2)\delta_m^*(\mathbf{k}').$$

Therefore

$$\langle \delta_g^s(\mathbf{k})\delta_g^{s*}(\mathbf{k}')\rangle = b^2(1+\beta\mu^2)(1+\beta\mu'^2)\langle \delta_m(\mathbf{k})\delta_m^*(\mathbf{k}')\rangle.$$

The Dirac delta enforces $\mathbf{k} = \mathbf{k}'$, so $\mu' = \mu$. Using

$$\langle \delta_m(\mathbf{k})\delta_m^*(\mathbf{k}')\rangle = (2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')P_m(k),$$

we get

$$\langle \delta_g^s(\mathbf{k})\delta_g^{s*}(\mathbf{k}')\rangle = (2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')\, b^2(1+\beta\mu^2)^2P_m(k).$$

By the definition of the redshift-space galaxy power spectrum,

$$\boxed{P_g^s(k,\mu) = b^2(1+\beta\mu^2)^2P_m(k).}$$

This is the Kaiser formula in the form used in the lecture notes.

Problem 3. Multipole coefficients

Question. Using the Kaiser formula, show that the monopole coefficient is

$$1 + \frac{2\beta}{3} + \frac{\beta^2}{5}.$$

You may use the integrals

$$\int_{-1}^{1}\mu^2\,\mathrm{d}\mu = \frac{2}{3}, \qquad \int_{-1}^{1}\mu^4\,\mathrm{d}\mu = \frac{2}{5}.$$

Solution. The monopole is the $\ell = 0$ Legendre coefficient. Since $\mathcal{L}_0(\mu) = 1$,

$$P_0(k) = \frac{1}{2}\int_{-1}^{1} P_g^s(k,\mu)\,\mathrm{d}\mu.$$

Substitute the Kaiser formula:

$$P_0(k) = \frac{1}{2}\int_{-1}^{1} b^2(1+\beta\mu^2)^2P_m(k)\,\mathrm{d}\mu.$$

Because $b^2P_m(k)$ does not depend on μ, we can pull it out of the integral:

$$P_0(k) = b^2P_m(k)\,\frac{1}{2}\int_{-1}^{1}(1+\beta\mu^2)^2\,\mathrm{d}\mu.$$

Now expand the square:

$$(1+\beta\mu^2)^2 = 1 + 2\beta\mu^2 + \beta^2\mu^4.$$

Hence

$$P_0(k) = b^2 P_m(k)\,\frac{1}{2}\int_{-1}^{1}\left(1 + 2\beta\mu^2 + \beta^2\mu^4\right)\mathrm{d}\mu.$$

Using

$$\int_{-1}^{1} 1\,\mathrm{d}\mu = 2, \qquad \int_{-1}^{1} \mu^2\,\mathrm{d}\mu = \frac{2}{3}, \qquad \int_{-1}^{1} \mu^4\,\mathrm{d}\mu = \frac{2}{5},$$

we obtain

$$P_0(k) = b^2 P_m(k)\,\frac{1}{2}\left(2 + 2\beta\cdot\frac{2}{3} + \beta^2\cdot\frac{2}{5}\right).$$

Simplifying,

$$P_0(k) = b^2\left(1 + \frac{2\beta}{3} + \frac{\beta^2}{5}\right)P_m(k).$$

Therefore the monopole coefficient is

$$\boxed{1 + \frac{2\beta}{3} + \frac{\beta^2}{5}.}$$

Problem 4. AP scaling

Question. If $\alpha_\perp = 1.03$ and $\alpha_\parallel = 0.97$, explain whether radial scales are inferred to be too large or too small in the fiducial cosmology.

Solution. The lecture notes define the AP scaling by

$$s_\perp^{\mathrm{fid}} = \alpha_\perp s_\perp, \qquad s_\parallel^{\mathrm{fid}} = \alpha_\parallel s_\parallel.$$

With $\alpha_\parallel = 0.97$, we have

$$s_\parallel^{\mathrm{fid}} = 0.97\, s_\parallel.$$

So the radial separation assigned in the fiducial cosmology is only 97% of the true radial separation. That means the fiducial cosmology makes line-of-sight distances appear *too small* by about 3%.

Thus the answer is

$$\boxed{\text{radial scales are inferred to be too small in the fiducial cosmology.}}$$

For completeness, $\alpha_\perp = 1.03$ means transverse scales are inferred to be too large by about 3%.

Problem 5. Kaiser versus Fingers-of-God

Question. In one page or less, compare the physical origin of the large-scale Kaiser effect and the small-scale Fingers-of-God effect.

Solution. The Kaiser effect and Fingers-of-God are both redshift-space distortions, but they arise in very different dynamical regimes.

Large-scale Kaiser effect. On large scales the peculiar velocity field is coherent. Galaxies fall toward overdense regions and flow out of underdense regions in an organized way. This coherent infall changes the observed redshift and therefore changes the inferred radial positions of galaxies. In configuration space, the large-scale clustering pattern tends to look squashed along the line of sight. In Fourier space, linear theory gives the anisotropic enhancement

$$P_g^s(k,\mu) = \left(b + f\mu^2\right)^2 P_m(k),$$

so the effect is directly tied to the growth rate f. Therefore the Kaiser effect is a *large-scale, linear, growth-sensitive* distortion.

Small-scale Fingers-of-God. Inside virialized halos, galaxies have large random motions rather than coherent bulk flows. A cluster may contain galaxies moving in many directions with substantial velocity dispersion. Because redshift converts velocity into line-of-sight position, this random motion smears the apparent galaxy distribution in the radial direction. In configuration space, compact objects become elongated along the line of sight, producing the familiar "fingers" pointing toward the observer. In Fourier space, these random velocities suppress power at large $k\mu$, and the effect is often modeled with a damping factor such as

$$D_{\rm FOG}(k\mu\sigma_v).$$

So Fingers-of-God are a *small-scale, nonlinear, velocity-dispersion* effect.

Comparison. The key contrast is the following:

- **Kaiser:** coherent infall on large scales, usually gives squashing, and probes the growth of structure.
- **Fingers-of-God:** random virial motion on small scales, usually gives elongation, and reflects halo-scale velocity dispersion.

Thus the two effects have the same observational origin in peculiar velocities, but they correspond to opposite physical limits: ordered bulk flow versus random internal motion.

Problem 6. Optional coding task (Julia)

Question. Plot the Kaiser anisotropy factor $(1+\beta\mu^2)^2$ as a function of μ for $\beta = 0.2$, 0.4, and 0.8.

Solution. A compact Julia solution is shown below. The same script is also saved separately as `lecture8_kaiser_factor.jl`.

```
using Plots

mus = range(-1.0, 1.0; length = 400)
betas = [0.2, 0.4, 0.8]

plt = plot(; xlabel = "mu", ylabel = "(1 + beta*mu^2)^2", linewidth = 2, legend = :topleft)
for beta in betas
    vals = (1 .+ beta .* mus.^2).^2
    plot!(plt, mus, vals; label = "beta = $(beta)", linewidth = 2)
end

savefig(plt, "lecture8_kaiser_factor.pdf")
```

Because the factor depends on μ^2, each curve is symmetric about $\mu = 0$. At $\mu = 0$ the value is always 1, while at $|\mu| = 1$ the value becomes $(1+\beta)^2$. So the three curves end at

$$(1+0.2)^2 = 1.44, \qquad (1+0.4)^2 = 1.96, \qquad (1+0.8)^2 = 3.24.$$

Therefore larger β produces stronger anisotropy between transverse modes ($\mu \approx 0$) and line-of-sight modes ($|\mu| \approx 1$).