Solutions to the Homework in Lecture 7
Fundamentals of Observational Cosmology (II)

These notes provide worked solutions to the homework problems in Lecture 7. Throughout, we use the main relations quoted in the lecture notes,

$$\Delta\theta \approx \frac{r_s}{D_M(z)}, \qquad \Delta z \approx \frac{H(z)\, r_s}{c},$$

and the standard Legendre expansion

$$\xi(s, \mu) = \sum_{\ell=0}^{\infty} \xi_\ell(s)\, \mathcal{L}_\ell(\mu).$$

Problem 1. Correlation function

Question. Starting from

$$\mathrm{d}P = \bar{n}^2[1 + \xi(\mathbf{r})]\mathrm{d}V_1\mathrm{d}V_2,$$

explain in words what it means for $\xi(r)$ to be positive, zero, or negative.

Solution. The quantity $\mathrm{d}P$ is the probability of finding one galaxy in volume element $\mathrm{d}V_1$ and another galaxy in volume element $\mathrm{d}V_2$, separated by $r = |\mathbf{r}|$. If galaxies were distributed completely at random with mean number density $\bar{n}$, we would have

$$\mathrm{d}P_{\mathrm{random}} = \bar{n}^2\, \mathrm{d}V_1\mathrm{d}V_2.$$

The factor $1 + \xi(r)$ therefore measures the *excess* or *deficit* of pairs relative to a random catalog.

- If $\xi(r) > 0$, then $1 + \xi(r) > 1$, so there are *more* pairs at separation r than in a random distribution. This means galaxies are positively correlated or clustered on that scale.
- If $\xi(r) = 0$, then $\mathrm{d}P = \bar{n}^2\mathrm{d}V_1\mathrm{d}V_2$, exactly the random expectation. This means there is no correlation at that separation.
- If $\xi(r) < 0$, then $1 + \xi(r) < 1$, so there are *fewer* pairs than in a random distribution. This means galaxies avoid one another on that scale, or are anti-correlated.

The limiting value $\xi(r) = -1$ would mean zero pair probability at that separation. Thus the sign of ξ tells us whether clustering is enhanced, random, or suppressed relative to Poisson sampling.

Problem 2. Multipole expansion

Question. Show that if $\xi(s,\mu)$ is symmetric under $\mu \to -\mu$, then only even Legendre multipoles appear in the expansion.

Solution. Write the Legendre expansion as

$$\xi(s,\mu) = \sum_{\ell=0}^{\infty} \xi_\ell(s)\, \mathcal{L}_\ell(\mu).$$

The coefficient of the ℓth multipole is

$$\xi_\ell(s) = \frac{2\ell+1}{2} \int_{-1}^{1} \xi(s,\mu)\, \mathcal{L}_\ell(\mu)\, \mathrm{d}\mu.$$

Now assume that

$$\xi(s,-\mu) = \xi(s,\mu),$$

so $\xi(s,\mu)$ is an even function of μ. Legendre polynomials satisfy the parity relation

$$\mathcal{L}_\ell(-\mu) = (-1)^\ell \mathcal{L}_\ell(\mu).$$

Therefore:

- if ℓ is even, $\mathcal{L}_\ell(\mu)$ is even;
- if ℓ is odd, $\mathcal{L}_\ell(\mu)$ is odd.

For odd ℓ, the integrand

$$\xi(s,\mu)\, \mathcal{L}_\ell(\mu)$$

is the product of an even function and an odd function, hence it is odd. The integral of an odd function from -1 to 1 vanishes, so

$$\xi_\ell(s) = 0 \qquad \text{for all odd } \ell.$$

Thus only even multipoles survive:

$$\boxed{\xi(s,\mu) = \xi_0(s)\mathcal{L}_0(\mu) + \xi_2(s)\mathcal{L}_2(\mu) + \xi_4(s)\mathcal{L}_4(\mu) + \cdots .}$$

This is exactly what happens for the usual anisotropic clustering signal, because reversing the line of sight leaves the physics unchanged.

Problem 3. BAO angular scale

Question. Assume a flat ΛCDM model with $\Omega_\mathrm{m} = 0.3$, $\Omega_\Lambda = 0.7$, and $D_M(z = 0.8) = 2800$ Mpc. Estimate the BAO angular scale $\Delta\theta$ using $r_s = 147$ Mpc.

Solution. The lecture notes give the approximate angular BAO scale as

$$\Delta\theta \approx \frac{r_s}{D_M(z)}.$$

Substituting the numbers,

$$\Delta\theta \approx \frac{147 \text{ Mpc}}{2800 \text{ Mpc}} = 0.0525.$$

This is in radians. Converting to degrees,

$$0.0525 \times \frac{180}{\pi} \approx 3.01^\circ.$$

Hence

$$\boxed{\Delta\theta \approx 0.0525 \text{ rad} \approx 3.0^\circ.}$$

So the BAO feature spans a few degrees on the sky at $z = 0.8$ for this choice of cosmology.

Problem 4. BAO radial scale

Question. Using the same cosmology, suppose $H(z = 0.8) = 109\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$. Estimate the BAO redshift interval Δz.

Solution. The radial BAO relation from the lecture is

$$\Delta z \approx \frac{H(z)\, r_s}{c}.$$

Using

$$H(z) = 109\ \mathrm{km\,s^{-1}\,Mpc^{-1}}, \qquad r_s = 147\ \mathrm{Mpc}, \qquad c \approx 2.998 \times 10^5\ \mathrm{km\,s^{-1}},$$

we obtain

$$\Delta z \approx \frac{109 \times 147}{299792.458} \approx 0.0534.$$

Therefore

$$\boxed{\Delta z \approx 0.053.}$$

So the BAO peak corresponds to a redshift separation of order five hundredths along the line of sight.

Problem 5. Conceptual comparison

Question. In one page or less, explain the physical difference between BAO, AP, and RSD. Your answer should state clearly which effect is mainly sensitive to geometry and which effect is mainly sensitive to growth.

Solution. BAO, AP, and RSD all affect the observed clustering of galaxies, but they come from different physics.

BAO. Baryon acoustic oscillations are a relic of sound waves in the tightly coupled photon–baryon plasma before recombination. Those waves imprint a preferred comoving scale, the sound horizon r_s, into the late-time matter distribution. In galaxy clustering this appears as a weak excess of pairs at a characteristic separation. Because the physical ruler is known, the observed angular and radial size of the BAO feature can be used to infer $D_M(z)$ and $H(z)$. So BAO is mainly a *geometric standard-ruler probe*, although nonlinear growth can broaden the feature and slightly affect its visibility.

AP. The Alcock–Paczynski effect is not a physical feature in the density field itself. It is a *geometric distortion* that arises if we convert observed angles and redshifts to distances using the wrong fiducial cosmology. A structure that is intrinsically isotropic can then appear stretched differently in the radial and transverse directions. This effect depends only on the distance–redshift relations used in the coordinate conversion, usually summarized by $D_M(z)$ and $H(z)$. Therefore AP is a *pure geometry test.*

RSD. Redshift-space distortions come from galaxy peculiar velocities. Observed redshift contains both the cosmological expansion and the Doppler contribution from motion along the line of sight. As a result, the inferred radial position is shifted. On large scales, coherent infall into overdense regions produces the Kaiser effect; on small scales, random virial motions inside halos produce Fingers-of-God. Because peculiar velocities are sourced by gravitational growth, RSD is mainly sensitive to the *growth of structure*, usually through parameters such as f or $f\sigma_8$.

Bottom line. In short:

- **BAO:** primordial standard ruler, mainly probes geometry through D_M and H;
- **AP:** apparent anisotropy from using the wrong cosmology, purely geometric;
- **RSD:** anisotropy from peculiar velocities, mainly probes growth and dynamics.

In real analyses these effects are fitted together because they all appear in the same anisotropic clustering pattern, but their physical origins are distinct.

Problem 6. Optional coding task (Julia)

Question. Make a sketch or simple plot showing how an intrinsically circular feature would look after an AP distortion with $\alpha_\perp \neq \alpha_\parallel$.

Solution. A simple way to visualize AP distortion is to start from a circle,

$$x^2 + y^2 = R^2,$$

and rescale the transverse and radial directions differently:

$$x' = \alpha_\perp x, \qquad y' = \alpha_\parallel y.$$

If $\alpha_\perp \neq \alpha_\parallel$, the image of the circle is an ellipse. For example, if $\alpha_\perp > \alpha_\parallel$, the feature looks stretched transversely and compressed radially.

Below is a compact Julia script that plots the original circle and the distorted ellipse. The same code is also saved separately as `lecture7_ap_distortion.jl`.

```
using Plots

R = 1.0
alpha_perp = 1.10
alpha_parallel = 0.90

phi = range(0, 2pi; length = 400)
x = R .* cos.(phi)
y = R .* sin.(phi)

x_ap = alpha_perp .* x
y_ap = alpha_parallel .* y

plt = plot(x, y;
    label = "true circle",
    linewidth = 2,
    aspect_ratio = :equal,
    xlabel = "transverse direction",
    ylabel = "line-of-sight direction")
plot!(plt, x_ap, y_ap; label = "after AP distortion", linewidth = 2, linestyle = :dash)

savefig(plt, "lecture7_ap_distortion.pdf")
```

The resulting figure should show a circle and an ellipse on the same axes. That is the simplest visual signature of anisotropic AP scaling.