LECTURE 6: FUNDAMENTALS OF OBSERVATIONAL COSMOLOGY (II)

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1. Bayes' Theorem

The Bayes' Theorem lays the foundation of Bayesian statistics, and it states,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{1}$$

- $P(A)$: The probability for event A to occur;
- $P(B)$: The probability for event B to occur;
- $P(A|B)$: The probability for event A to occur, under the condition that event B has occurred;
- $P(B|A)$: The probability for event B to occur, under the condition that event A has occurred.

An easy proof:

$$P(B|A)P(A) = P(B \cap A) = P(A \cap B) = P(A|B)P(B) \tag{2}$$

The Bayes' Theorem has wide applications. For data analysis, we get,

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)} \tag{3}$$

where D denotes observational data, and θ for a set of cosmological parameters.

- $P(\theta|D)$: The posterior distribution;
- $P(D|\theta)$: The liklihood function;
- $P(\theta)$: The prior (it could be flat, or Gaussian);
- $P(D)$: The Evidence, usually is used as a trivial normalisation factor.

An example:

A mobile phone produced by The XPhone Company was found to be defective (D). There are three factories (A, B, C) where such phones are manufactured. A Quality Control Manager (QCM) is responsible for investigating the source of found defects. This is what the QCM knows about the company's phone production and the possible source of defects:

The QCM would like to answer the following question: If a randomly selected phone is defective, what is the probability that the phone was manufactured in factory A, B and C, respectively?

Factory	% of total production	Probability of phones
A	$0.35 = P(A)$	$0.015 = P(D\|A)$
B	$0.35 = P(B)$	$0.010 = P(D\|B)$
C	$0.30 = P(C)$	$0.020 = P(D\|C)$

Solution:

$$P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{0.015 \times 0.35}{0.015.35 + 0.010 \times 0.35 + 0.020 \times 0.30} = 0.356 \tag{4}$$

$$P(B|D) = \frac{P(D|B)P(B)}{P(D)} = \frac{0.010 \times 0.35}{0.015.35 + 0.010 \times 0.35 + 0.020 \times 0.30} = 0.237 \tag{5}$$

$$P(C|D) = \frac{P(D|C)P(C)}{P(D)} = \frac{0.020 \times 0.30}{0.015.35 + 0.010 \times 0.35 + 0.020 \times 0.30} = 0.407 \tag{6}$$

2. Mean, variance and covariance

$$\mu = \frac{1}{N}\sum_{i=1}^{N} X_i \tag{7}$$

$$\sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \mu)^2 \tag{8}$$

$$\mathrm{Cov}(X,Y) = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \mu_X)(Y_i - \mu_Y) \tag{9}$$

$$\mathrm{Corr}(X,Y) = \frac{1}{N-1}\sum_{i=1}^{N}\frac{(X_i - \mu_X)(Y_i - \mu_Y)}{\sigma_X \sigma_Y} \tag{10}$$

3. The central limit theorem

In probability theory, the central limit theorem (CLT) establishes that, in some situations, when independent random variables are added, their properly normalized sum tends toward a normal distribution even if the original variables themselves are not normally distributed.

4. Gaussian distribution

$$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{11}$$

$$f_{\boldsymbol{x}}(x_1, ..., x_k) = \frac{\exp\left[-\frac{1}{2}(\boldsymbol{x}-\mu)^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\mu)\right]}{\sqrt{(2\pi)^k |\boldsymbol{\Sigma}|}} \tag{12}$$

5. χ^2 fitting

$$\chi^2 = -2\ln \mathcal{L} = (\boldsymbol{x}-\mu)^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\mu) \tag{13}$$

6. The 68, 95, 99% rule

$$\frac{\int_0^1 e^{-x^2/2}\mathrm{d}x}{\int_0^\infty e^{-x^2/2}\mathrm{d}x} = 68.3\%; \quad \frac{\int_0^2 e^{-x^2/2}\mathrm{d}x}{\int_0^\infty e^{-x^2/2}\mathrm{d}x} = 95.5\%; \quad \frac{\int_0^3 e^{-x^2/2}\mathrm{d}x}{\int_0^\infty e^{-x^2/2}\mathrm{d}x} = 99.7\%; \tag{14}$$

7. General cases

χ^2 distribution:

$$f(x;k) = \frac{x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{k}{2}} \Gamma(\frac{k}{2})} (x > 0) \tag{15}$$

Cumulative:

$$F(x;k) = \frac{\gamma(\frac{k}{2}, \frac{x}{2})}{\Gamma(\frac{k}{2})} \tag{16}$$

where $\gamma(s,t)$ is the lower incomplete gamma function.

Special cases:

$$F(x;1) = \mathrm{erf}(\sqrt{x}) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\mathrm{d}t \tag{17}$$

$$F(x;2) = 1 - e^{-x/2} \tag{18}$$

So for 2-variables, $\Delta\chi^2 = 2.31, 6.18, 11.83$ for $F = 68,\ 95,\ 99\%$.

References

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[3] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, "Numerical Recipes in FORTRAN: The Art of Scientific Computing."