Solutions to the Homework in Lecture 5
Fundamentals of Observational Cosmology (I)

These notes provide worked solutions to the homework problems in Lecture 5. Throughout, we use

$$E(z) = \frac{H(z)}{H_0}, \qquad D_L(z) = (1+z)D_M(z), \qquad \mu = 5\log_{10}\left(\frac{D_L}{\mathrm{Mpc}}\right) + 25.$$

Problem 1. Low-redshift limit

Question. Show that for sufficiently small z,

$$D_L(z) \approx \frac{cz}{H_0}.$$

Use this to derive the leading small-z form of the distance modulus.

Solution. For very small redshift, the expansion history has not changed much over the photon travel time, so

$$E(z) = 1 + \mathcal{O}(z).$$

Hence the line-of-sight comoving distance is

$$D_C(z) = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{E(z')} \approx \frac{c}{H_0}\int_0^z \mathrm{d}z' = \frac{cz}{H_0}.$$

At the same order, curvature effects are higher order in z, so $D_M \approx D_C$. Therefore

$$D_L(z) = (1+z)D_M(z) \approx (1+z)\frac{cz}{H_0}.$$

To leading order, $(1+z) = 1 + \mathcal{O}(z)$, so the dominant term is

$$\boxed{D_L(z) \approx \frac{cz}{H_0}.}$$

This is the luminosity-distance version of the Hubble law.

Now substitute this into the definition of the distance modulus:

$$\mu \approx 5\log_{10}\left(\frac{cz/H_0}{\mathrm{Mpc}}\right) + 25.$$

Equivalently,

$$\boxed{\mu(z) \approx 5\log_{10} z + 5\log_{10}\left(\frac{c/H_0}{\mathrm{Mpc}}\right) + 25.}$$

So at low redshift the Hubble diagram is approximately a straight line in m versus $\log_{10} z$. For example, if $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$, then $c/H_0 \approx 4282.75$ Mpc, giving

$$\mu(z) \approx 5\log_{10} z + 43.16.$$

As $z \to 0$, $D_L \to 0$ and therefore $\mu \to -\infty$, which is why numerical plots of $\mu(z)$ should not include the exact point $z = 0$.

Problem 2. Numerical calculation

Question. Assume a flat ΛCDM model with $\Omega_{\rm m} = 0.3$, $\Omega_\Lambda = 0.7$, and $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$. Evaluate $D_L(z)$ and $\mu(z)$ at $z = 0.5$ and $z = 1.0$.

Solution. For a flat model, $\Omega_k = 0$, so

$$D_L(z) = (1+z)\frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{\sqrt{0.3(1+z')^3 + 0.7}}.$$

We take $c = 299792.458\ \mathrm{km\,s^{-1}}$ and $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$, so

$$\frac{c}{H_0} = 4282.75\ \mathrm{Mpc}.$$

At $z = 0.5$

Numerically,

$$I(0.5) \equiv \int_0^{0.5} \frac{\mathrm{d}z'}{\sqrt{0.3(1+z')^3 + 0.7}} \approx 0.440984.$$

Therefore

$$D_L(0.5) = 1.5 \times 4282.75 \times 0.440984\ \mathrm{Mpc} \approx 2832.94\ \mathrm{Mpc} \approx 2.83\ \mathrm{Gpc}.$$

The distance modulus is then

$$\mu(0.5) = 5\log_{10}(2832.94) + 25 \approx 42.261.$$

So

$$\boxed{D_L(0.5) \approx 2.83\ \mathrm{Gpc}, \qquad \mu(0.5) \approx 42.26.}$$

At $z = 1.0$

Similarly,

$$I(1.0) \equiv \int_0^{1.0} \frac{\mathrm{d}z'}{\sqrt{0.3(1+z')^3 + 0.7}} \approx 0.771427.$$

Hence

$$D_L(1.0) = 2.0 \times 4282.75 \times 0.771427\ \mathrm{Mpc} \approx 6607.66\ \mathrm{Mpc} \approx 6.61\ \mathrm{Gpc}.$$

Then

$$\mu(1.0) = 5\log_{10}(6607.66) + 25 \approx 44.100.$$

Therefore

$$\boxed{D_L(1.0) \approx 6.61\ \mathrm{Gpc}, \qquad \mu(1.0) \approx 44.10.}$$

Problem 3. Coding exercise (Julia)

Question. Write a short program in Fortran, Julia, or Python that computes $\mu(z)$ on a grid from $z = 0$ to 1.5 for three models: $(\Omega_{\rm m}, \Omega_\Lambda) = (1, 0)$, $(0.3, 0.7)$, and $(0.3, 0)$. Plot the three curves on the same figure and briefly describe the differences.

Solution. Below is a compact Julia solution. It uses a simple composite Simpson rule for the integral and the `Plots` package for the figure. Because $\mu(0)$ is undefined, the grid starts at a small positive value instead of exactly zero.

```
using Printf
using Plots

const c = 299792.458   # km/s
const H0 = 70.0        # km s^-1 Mpc^-1

E(z, Om, Ode) = sqrt(Om * (1 + z)^3 + Ode + (1 - Om - Ode) * (1 + z)^2)

function simpson(f, a, b; n = 2000)
    n = iseven(n) ? n : n + 1
    h = (b - a) / n
    s = f(a) + f(b)
    for i in 1:2:n-1
        s += 4 * f(a + i * h)
    end
    for i in 2:2:n-2
        s += 2 * f(a + i * h)
    end
    return h * s / 3
end

function D_C(z, Om, Ode; H0 = H0, n = 2000)
    integral = simpson(zp -> 1 / E(zp, Om, Ode), 0.0, z; n = n)
    return (c / H0) * integral
end

function D_M(z, Om, Ode; H0 = H0, n = 2000)
    Ok = 1.0 - Om - Ode
    Dc = D_C(z, Om, Ode; H0 = H0, n = n)
    if abs(Ok) < 1e-12
        return Dc
    elseif Ok > 0
        x = sqrt(Ok) * H0 * Dc / c
        return c / (H0 * sqrt(Ok)) * sinh(x)
    else
        x = sqrt(-Ok) * H0 * Dc / c
        return c / (H0 * sqrt(-Ok)) * sin(x)
    end
end

D_L(z, Om, Ode; H0 = H0, n = 2000) = (1 + z) * D_M(z, Om, Ode; H0 = H0, n = n)
mu(z, Om, Ode; H0 = H0, n = 2000) = 5 * log10(D_L(z, Om, Ode; H0 = H0, n = n)) + 25

for z in (0.5, 1.0)
    dl = D_L(z, 0.3, 0.7)
    @printf("z = %.1f: D_L = %.3f Mpc, mu = %.3f\n", z, dl, mu(z, 0.3, 0.7))
end

zs = range(1e-3, 1.5; length = 250)
models = [
    ("EdS (Om=1, Ode=0)", 1.0, 0.0),
    ("flat LCDM (Om=0.3, Ode=0.7)", 0.3, 0.7),
    ("open matter-only (Om=0.3, Ode=0)", 0.3, 0.0)
]

plt = plot(xlabel = "z", ylabel = "mu(z)", linewidth = 2, legend = :topleft)
for (label, Om, Ode) in models
    plot!(plt, zs, [mu(z, Om, Ode) for z in zs], label = label)
end
savefig(plt, "lecture5_mu_curves.pdf")
```

Running the code produces the expected Hubble-diagram ordering. Representative numerical values are:

z	$\mu_{(1,0)}$	$\mu_{(0.3,0.7)}$	$\mu_{(0.3,0)}$
0.5	41.862	42.261	42.046
1.0	43.502	44.100	43.844
1.5	44.480	45.189	44.955

Description of the curves. At very low redshift, all three curves are nearly identical because they all reduce to the Hubble law. At higher redshift, the Einstein-de Sitter model $(\Omega_\mathrm{m}, \Omega_\Lambda) = (1, 0)$ gives the smallest distance modulus, the open matter-only model $(0.3, 0)$ lies in between, and the flat ΛCDM model $(0.3, 0.7)$ gives the largest distance modulus. The reason is physical: stronger deceleration in the past makes objects appear closer today, while a cosmological constant produces late-time acceleration and therefore larger luminosity distances at fixed redshift. That is exactly why supernova Hubble diagrams are sensitive to accelerated expansion.

Problem 4. Interpretation

Question. In one page or less, explain why supernova data alone do not directly determine both H_0 and the absolute magnitude M without additional calibration information.

Solution. A supernova observation gives an apparent magnitude m at a measured redshift z. The theoretical relation is

$$m(z) = M + \mu(z) = M + 5\log_{10}\left(\frac{D_L(z)}{\mathrm{Mpc}}\right) + 25.$$

Now the luminosity distance always contains an overall factor of H_0^{-1}. Schematically one can write

$$D_L(z) = \frac{c}{H_0}\, d_L(z; \Omega_m, \Omega_\Lambda, \ldots),$$

where d_L is a dimensionless function that describes the *shape* of the Hubble diagram. Substituting this into the magnitude relation gives

$$m(z) = 5\log_{10} d_L(z; \Omega_m, \Omega_\Lambda, \ldots) + \Big[M - 5\log_{10} H_0 + \text{constant}\Big].$$

The key point is that the data constrain only the *combination*

$$M - 5\log_{10} H_0,$$

not M and H_0 separately. If one increases H_0, all model distances shrink by the same factor. That change can be exactly compensated by choosing a brighter absolute magnitude M. So the supernova Hubble diagram by itself fixes relative distances as a function of redshift, but not the absolute distance scale.

This is why supernova analyses usually introduce a nuisance parameter such as $\mathcal{M}$ that absorbs the combination of M and H_0. With supernovae alone one can constrain the *shape* of the expansion history very well, for example parameters like Ω_m or the presence of late-time acceleration, because those parameters change how the curve bends with redshift. But one cannot extract a unique value of H_0 unless the absolute luminosity scale is calibrated independently.

Such calibration comes from additional distance-ladder information, for example Cepheids, the tip of the red giant branch, masers, or other standard candles and rulers that anchor the absolute magnitude of Type Ia supernovae. Once M is calibrated, the degeneracy is broken and the same supernova data can then be used to infer H_0. Without that external calibration, supernovae are best thought of as superb *relative* distance indicators rather than fully absolute ones.