Solutions to the Homework in Lecture 4
Fundamentals of General Relativity (II)

These notes provide worked solutions to the homework problems in Lecture 4. We use dots for derivatives with respect to cosmic time t and primes for derivatives with respect to conformal time τ.

Problem 1. Conformal-time practice

Question. Starting from $\mathrm{d}\tau = \mathrm{d}t/a$, show explicitly that $\mathcal{H} = aH$. Then derive $\mathrm{d}/\mathrm{d}\tau = a\,\mathrm{d}/\mathrm{d}t$.

Solution. By definition,

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{a(t)}.$$

Taking reciprocals gives

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = a.$$

Now apply the chain rule to the scale factor:

$$a' \equiv \frac{\mathrm{d}a}{\mathrm{d}\tau} = \frac{\mathrm{d}a}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \dot{a}\,a.$$

Therefore

$$\mathcal{H} \equiv \frac{a'}{a} = \frac{a\dot{a}}{a} = \dot{a}.$$

Since the usual Hubble parameter is

$$H \equiv \frac{\dot{a}}{a},$$

we immediately obtain

$$\boxed{\mathcal{H} = aH.}$$

For any time-dependent quantity $X(t)$,

$$X' = \frac{\mathrm{d}X}{\mathrm{d}\tau} = \frac{\mathrm{d}X}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = a\frac{\mathrm{d}X}{\mathrm{d}t}.$$

Hence the differential operators satisfy

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau} = a\frac{\mathrm{d}}{\mathrm{d}t}.}$$

This is one of the most useful conversion rules in perturbation theory.

Problem 2. Anisotropic stress

Question. Read the meaning of anisotropic stress and explain why a perfect fluid gives $\Phi = \Psi$ at linear order.

Solution. Anisotropic stress is the part of the stress tensor that depends on direction. In other words, it is the traceless spatial part of the perturbed energy-momentum tensor. If the stress is the same in every spatial direction, then there is no anisotropic stress. If different directions experience different stresses, or if the medium supports shear, then anisotropic stress is present.

For a perfect fluid,

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}.$$

Its spatial stress is purely isotropic: in the fluid rest frame the spatial part is just $P\delta^i{}_j$. There is no shear term and no traceless directional stress. Therefore the scalar anisotropic-stress perturbation vanishes.

At linear order, the traceless part of the spatial Einstein equations relates the difference of the two Newtonian-gauge potentials to anisotropic stress. A standard form is

$$k^2(\Phi - \Psi) = 12\pi G a^2(\bar{\rho} + \bar{P})\sigma,$$

where σ is the scalar shear stress. Equivalently, one may write the same statement in terms of a traceless stress tensor $\Pi^i{}_j$. The important physical point is the same:

no anisotropic stress $\Longrightarrow$ no source for $\Phi - \Psi$.

Hence for a perfect fluid,

$$\sigma = 0 \quad \Longrightarrow \quad \boxed{\Phi = \Psi.}$$

Physical interpretation. The potential Ψ perturbs the time-time part of the metric and behaves like a Newtonian gravitational potential. The potential Φ describes spatial curvature perturbations. If the matter sector has no directional shear, general relativity ties these two scalar perturbations together so that they are equal.

Examples. Photons and free-streaming neutrinos can generate anisotropic stress, so in those cases one can have $\Phi \neq \Psi$. But for the idealized perfect-fluid matter treatment used in introductory linear theory, the standard result is $\Phi = \Psi$.

Problem 3. Growth equation

Question. Starting from

$$\delta_m' = -\theta_m + 3\Phi', \qquad \theta_m' + \mathcal{H}\theta_m = k^2\Psi,$$

and using the sub-horizon approximation with $\Phi = \Psi$, derive the matter-growth equation in cosmic time:

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\,\delta_m = 0.$$

Solution. On sub-horizon scales, $k \gg \mathcal{H}$, the metric potentials vary slowly compared with the matter flow. In that regime one neglects the $3\Phi'$ term in the continuity equation and writes

$$\delta_m' \simeq -\theta_m.$$

Differentiate once more with respect to conformal time:

$$\delta_m'' \simeq -\theta_m'.$$

Now use the Euler equation,

$$\theta_m' = k^2\Psi - \mathcal{H}\theta_m.$$

Substituting this into the previous expression gives

$$\delta_m'' = -k^2\Psi + \mathcal{H}\theta_m.$$

Using $\theta_m \simeq -\delta_m'$ then yields

$$\delta_m'' + \mathcal{H}\delta_m' = -k^2\Psi.$$

To replace Ψ by δ_m, use the sub-horizon Poisson equation. In real space,

$$\nabla^2\Psi = 4\pi G a^2 \bar{\rho}_m \, \delta_m.$$

With the standard Fourier rule $\nabla^2 \to -k^2$, this becomes

$$-k^2\Psi = 4\pi G a^2 \bar{\rho}_m \, \delta_m.$$

Therefore

$$\delta_m'' + \mathcal{H}\delta_m' - 4\pi G a^2 \bar{\rho}_m \, \delta_m = 0.$$

This is the conformal-time form of the linear growth equation.

We now convert it to cosmic time. From Problem 1,

$$\delta_m' = a\dot{\delta}_m.$$

Differentiating once more,

$$\delta_m'' = \frac{\mathrm{d}}{\mathrm{d}\tau}(a\dot{\delta}_m) = a'\dot{\delta}_m + a\frac{\mathrm{d}\dot{\delta}_m}{\mathrm{d}\tau}.$$

Since $a' = a\mathcal{H} = a^2 H$ and $\mathrm{d}/\mathrm{d}\tau = a\,\mathrm{d}/\mathrm{d}t$,

$$\delta_m'' = a^2 H\dot{\delta}_m + a^2\ddot{\delta}_m = a^2(\ddot{\delta}_m + H\dot{\delta}_m).$$

Also,

$$\mathcal{H}\delta_m' = (aH)(a\dot{\delta}_m) = a^2 H\dot{\delta}_m.$$

Substituting into the conformal-time equation gives

$$a^2(\ddot{\delta}_m + H\dot{\delta}_m) + a^2 H\dot{\delta}_m - 4\pi G a^2 \bar{\rho}_m \, \delta_m = 0.$$

Divide by a^2:

$$\boxed{\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G \bar{\rho}_m \, \delta_m = 0.}$$

This is the standard growth equation for pressureless matter in general relativity.

Remark on signs. Different Fourier-transform conventions move a minus sign between the Poisson equation and the definition of k^2. The final physical growth equation above is convention independent.

Problem 4. Growing mode

Question. In an Einstein-de Sitter Universe, substitute $\delta_m \propto a^n$ into the growth equation and show that the growing mode has $n = 1$.

Solution. In an Einstein-de Sitter Universe,

$$\Omega_m = 1, \qquad \Omega_\Lambda = 0,$$

so the expansion is entirely matter dominated. The growth equation is

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\rho}_m\,\delta = 0.$$

Assume a power-law solution

$$\delta = Aa^n,$$

with constant A and exponent n. Then

$$\dot{\delta} = nH\delta,$$

because $\dot{a} = Ha$. Differentiating again,

$$\ddot{\delta} = n\dot{H}\,\delta + nH\dot{\delta} = n\dot{H}\,\delta + n^2H^2\delta.$$

Hence the growth equation becomes

$$\left[n\dot{H} + n^2H^2 + 2nH^2 - 4\pi G\bar{\rho}_m\right]\delta = 0.$$

For Einstein-de Sitter,

$$H^2 = \frac{8\pi G}{3}\bar{\rho}_m \quad\Longrightarrow\quad 4\pi G\bar{\rho}_m = \frac{3}{2}H^2.$$

Also, because $a \propto t^{2/3}$, one has

$$H = \frac{2}{3t}, \qquad \dot{H} = -\frac{2}{3t^2} = -\frac{3}{2}H^2.$$

Substituting these relations gives

$$\left[n\left(-\frac{3}{2}H^2\right) + n^2H^2 + 2nH^2 - \frac{3}{2}H^2\right]\delta = 0.$$

Factor out $H^2\delta$:

$$\left(n^2 + \frac{1}{2}n - \frac{3}{2}\right)H^2\delta = 0.$$

Therefore n satisfies

$$n^2 + \frac{1}{2}n - \frac{3}{2} = 0.$$

Multiplying by 2,

$$2n^2 + n - 3 = 0.$$

This factors as

$$(2n+3)(n-1) = 0.$$

So the two solutions are

$$\boxed{n = 1 \quad \text{or} \quad n = -\frac{3}{2}.}$$

The growing mode is therefore

$$\boxed{\delta_m \propto a,}$$

while the decaying mode scales as

$$\delta_m \propto a^{-3/2}.$$

This is the standard result for matter domination.