Lecture 4: Fundamentals of General Relativity (II)

Gong-Bo Zhao

Abstract

This lecture extends the background GR equations to linear perturbation theory, which is the starting point for modern cosmological observables and for Boltzmann codes such as CAMB. We focus on the physical meaning of perturbations, conformal-time notation, gauge choices, and the origin of the linear growth equation.

Learning goals

After this lecture, students should be able to:

- switch correctly between cosmic time and conformal time;
- explain the difference between background quantities and perturbations;
- write the scalar-perturbed metric in Newtonian gauge;
- identify the linearized Einstein and fluid equations used in structure formation;
- derive the matter growth equation on sub-horizon scales.

1 Why perturbation theory?

The background FLRW Universe is homogeneous and isotropic, but the real Universe contains small fluctuations in density, velocity, and gravitational potential. Cosmological perturbation theory studies these fluctuations when their amplitudes are still small:

$$\delta(\mathbf{x}, t) \equiv \frac{\rho(\mathbf{x}, t) - \bar{\rho}(t)}{\bar{\rho}(t)}, \qquad |\delta| \ll 1.$$

This linear regime is crucial because it connects early-Universe initial conditions to the CMB and to large-scale structure.

2 Conformal time conventions

In cosmological perturbation theory, it is common to use conformal time τ rather than cosmic time t:

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{a(t)}.$$

It is best to reserve:

- dots for derivatives with respect to cosmic time t;
- primes for derivatives with respect to conformal time τ.

Then

$$\mathcal{H} \equiv \frac{a'}{a} = aH.$$

This relation is exact and extremely useful in code and in analytic work.

3 Background equations in conformal time

The Friedmann equations can be written in conformal-time form as

$$\mathcal{H}^2 + k = \frac{8\pi G}{3} a^2 \bar{\rho},$$

and

$$\mathcal{H}' - \mathcal{H}^2 = -\frac{4\pi G}{3} a^2 (\bar{\rho} + 3\bar{P}).$$

These are equivalent to the usual cosmic-time equations, but often easier to combine with perturbation equations.

4 Metric perturbations

The most general perturbation of the metric can be decomposed into scalar, vector, and tensor parts. For structure formation, scalar perturbations are the most important. In conformal Newtonian gauge, the scalar-perturbed metric is

$$\mathrm{d}s^2 = a^2(\tau) \left[-(1 + 2\Psi)\mathrm{d}\tau^2 + (1 - 2\Phi)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j \right].$$

Here:

- Ψ plays the role of a gravitational potential in the time-time part;
- Φ describes spatial curvature perturbations.

In a perfect fluid with negligible anisotropic stress,

$$\Phi = \Psi.$$

This equality is widely used in introductory cosmology.

5 Linear perturbation equations

At linear order, the Einstein equations become a set of coupled differential equations for the metric and fluid perturbations. For scalar modes in Fourier space, one important constraint equation is the generalized Poisson equation:

$$k^2\Phi + 3\mathcal{H}\left(\Phi' + \mathcal{H}\Psi\right) = 4\pi G a^2 \delta\rho.$$

On sub-horizon scales ($k \gg \mathcal{H}$), the first term dominates, giving the familiar approximation

$$k^2\Phi \approx 4\pi G a^2 \delta\rho.$$

For pressureless matter, the linear continuity and Euler equations in Newtonian gauge can be written as

$$\delta_m' = -\theta_m + 3\Phi',$$

$$\theta_m' + \mathcal{H}\theta_m = k^2\Psi,$$

where θ_m is the velocity-divergence perturbation.

6 The matter growth equation

Combining the continuity equation, Euler equation, and sub-horizon Poisson equation gives the standard linear growth equation for nonrelativistic matter:

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\,\delta_m = 0.$$

This is the correct cosmic-time form. In a matter-dominated Einstein-de Sitter Universe, the growing solution is

$$\delta_m \propto a,$$

while the decaying solution scales as $a^{-3/2}$.

This result is one of the central facts of structure formation: during matter domination, density perturbations grow efficiently.

7 Growth rate and cosmological tests

A useful observable is the logarithmic growth rate

$$f(a) \equiv \frac{\mathrm{d}\ln\delta_m}{\mathrm{d}\ln a}.$$

In standard GR with matter domination, $f = 1$. At later times, dark energy slows the growth, and in modified-gravity models the relation between geometry and growth can change.

This is why combining distance probes (such as supernovae and BAO) with growth probes (such as redshift-space distortions) is so powerful.

8 Gauge choice and CAMB

Different gauges reorganize the same physical perturbations in different ways. CAMB mainly uses synchronous-gauge variables, while many analytic introductions use Newtonian gauge because the physical interpretation is more transparent.

The key lesson is:

> Observable predictions are gauge invariant, but intermediate variables in the calculation are not.

So when reading papers or code, always check the gauge convention before comparing formulas.

9 Summary

This lecture adds linear fluctuations on top of the homogeneous background. The essential steps are:

- use conformal time, with $\mathcal{H} = aH$;
- perturb the metric and matter fields around the FLRW background;
- solve the linearized Einstein and fluid equations;
- derive the growth of matter perturbations, with $\delta_m \propto a$ in matter domination.

This framework is the bridge between early-Universe initial conditions and late-time observables.

Suggested reading

- Ma and Bertschinger (1995), especially the discussion of gauges and perturbation variables.
- Dodelson and Schmidt, chapters on linear perturbation theory.

Homework

1. **Conformal-time practice.** Starting from $\mathrm{d}\tau = \mathrm{d}t/a$, show explicitly that $\mathcal{H} = aH$. Then derive $\mathrm{d}/\mathrm{d}\tau = a\,\mathrm{d}/\mathrm{d}t$.
2. **Anisotropic stress.** Read the meaning of anisotropic stress and explain why a perfect fluid gives $\Phi = \Psi$ at linear order.
3. **Growth equation.** Starting from
$$\delta_m' = -\theta_m + 3\Phi', \qquad \theta_m' + \mathcal{H}\theta_m = k^2\Psi,$$
and using the sub-horizon approximation with $\Phi = \Psi$, derive the matter-growth equation in cosmic time:
$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\,\delta_m = 0.$$
4. **Growing mode.** In an Einstein-de Sitter Universe, substitute $\delta_m \propto a^n$ into the growth equation and show that the growing mode has $n = 1$.