LECTURE 3: FUNDAMENTALS OF GENERAL RELATIVITY (I)

GONG-BO ZHAO

1. Basics of tensor analysis

$$A^\mu \equiv (A^0, A^i), (i = 1, 2, 3) \tag{1}$$

$$A_\mu = g_{\mu\nu}A^\nu; \quad A^\mu = g^{\mu\nu}A_\nu; \quad P_\mu P^\mu = P^2 = g_{\mu\nu}P^\mu P^\nu \tag{2}$$

So the metric can be used to lower or raise indices, even for itself, *e.g.*,

$$g^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}g_{\alpha\beta} \tag{3}$$

Set $\alpha = \nu$, we have,

$$g^{\nu\beta}g_{\alpha\beta} = \delta^\nu_\alpha \tag{4}$$

Note that for the FRW metric, the i-i component of $g_{\mu\nu}$ is a^2, but it is a^{-2} for $g^{\mu\nu}$.

2. The geodesic and the affine connection

The straight line in curves space is called a geodesic, whose equation is,

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0 \tag{5}$$

where the affine connection Γ can be evaluated using the metric, namely,

$$\Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right) \tag{6}$$

The connection takes care of the change of the unit vectors when moving from one place to another for an infinitesimal distance on a manifold, so it vanishes if there is no curvature.

Let us calculate the nonzero connections for the FRW metric,

$$\Gamma^0_{\alpha\beta} = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial t} \tag{7}$$

So,

$$\Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^0_{i0} = 0 \tag{8}$$

$$\Gamma^0_{ij} = a\dot{a}\ \delta_{ij} \tag{9}$$

$$\Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{a}\ \delta_{ij} \tag{10}$$

3. The Einstein equation

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}; \quad R \equiv g^{\mu\nu}R_{\mu\nu} \tag{11}$$

$$R_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} \tag{12}$$

For the background FRW metric,

$$R_{00} = -3\frac{\ddot{a}}{a}; \quad R_{ij} = \left(2\dot{a}^2 + a\ddot{a}\right)\delta_{ij}; \quad R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] \tag{13}$$

The 0-0 component of the Einstein equation yields,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{14}$$

and the i-i component gives us,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{15}$$

So the cosmic acceleration ($\ddot{a} > 0$) means that there exists energy component with a negative pressure!

4. Energy conservation

The energy conservation is mathematically equivalent to a vanishing covariant derivative of the energy-momentum tensor, namely,

$$\nabla_\mu T^\mu_\nu = \frac{\partial T^\mu_\nu}{\partial x^\mu} + \Gamma^\mu_{\alpha\mu}T^\alpha_\nu - \Gamma^\alpha_{\nu\mu}T^\mu_\alpha \tag{16}$$

It gives us the continuity equation (set $\nu = 0$), say,

$$\dot{\rho} + 3H\rho(1 + w) = 0; \quad w \equiv \frac{P}{\rho} \tag{17}$$

If w is a constant, then,

$$\rho \propto a^{-3(1+w)} \tag{18}$$

So for matter ($w = 0$), radiation ($w = 1/3$) and vacuum energy ($w = -1$), their energy densities evolves with a^{-3}, a^{-4} and a^0, respectively.

The Friedmann equation reads,

$$H^2(a) = H_0^2 \left[\Omega_{\rm M} a^{-3} + \Omega_{\rm R} a^{-4} + \Omega_{\rm K} a^{-2} + \Omega_{\rm X} X(a)\right] \tag{19}$$

$$X(a) \equiv \text{Exp}\left[-3 \int_1^a \frac{1 + w(a')}{a'} \text{d}a'\right] \tag{20}$$

where $\Omega_{\rm M} + \Omega_{\rm R} + \Omega_{\rm K} + \Omega_{\rm X} = 1$.

5. Homework

Ex. 2.2 and 2.6 of Dodelson & Schmidt [1] (page 53-54), and prove Eq (19).

References

[1] Dodelson, S. & Schmidt, F. 2020, Modern cosmology, Academic Press. ISBN 9780128159484, 2020, 512 p.