Solutions to the Homework in Lecture 3
Fundamentals of General Relativity (I)

These notes provide worked solutions to the homework problems in Lecture 3. We adopt the metric signature $(-,+,+,+)$ and use $c = 1$.

Problem 1. Index gymnastics

Question. Starting from $g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}{}_{\nu}$, show explicitly how one raises and lowers the indices of a vector and of the metric itself.

Solution. The relation

$$g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}{}_{\nu}$$

means that $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. This is exactly what allows indices to be moved up and down.

Lowering a vector index

Given a contravariant vector A^{μ}, define

$$A_{\mu} \equiv g_{\mu\nu}A^{\nu}.$$

This is the covariant version of the same vector.

Raising a vector index

Now apply $g^{\alpha\mu}$ to A_{μ}:

$$g^{\alpha\mu}A_{\mu} = g^{\alpha\mu}g_{\mu\nu}A^{\nu} = \delta^{\alpha}{}_{\nu}A^{\nu} = A^{\alpha}.$$

So raising the index gives back the original vector:

$$A^{\mu} = g^{\mu\nu}A_{\nu}.$$

Therefore the metric and inverse metric act as the operators that lower and raise indices:

$$\boxed{A_{\mu} = g_{\mu\nu}A^{\nu}, \qquad A^{\mu} = g^{\mu\nu}A_{\nu}.}$$

Raising the indices of the metric itself

Start from $g_{\alpha\beta}$ and raise both indices:

$$g^{\mu\alpha}g^{\nu\beta}g_{\alpha\beta}.$$

Using the inverse-metric relation, this is exactly the object with both indices up, namely

$$\boxed{g^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}g_{\alpha\beta}.}$$

Similarly, starting from $g^{\alpha\beta}$ and lowering both indices gives

$$\boxed{g_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}g^{\alpha\beta}.}$$

Useful FRW example

For the flat FRW metric,

$$g_{00} = -1, \qquad g_{0i} = 0, \qquad g_{ij} = a^2\delta_{ij},$$

and

$$g^{00} = -1, \qquad g^{0i} = 0, \qquad g^{ij} = a^{-2}\delta^{ij}.$$

So for any vector A^μ,

$$A_0 = g_{00}A^0 = -A^0, \qquad A_i = g_{ij}A^j = a^2\delta_{ij}A^j.$$

Conversely,

$$A^0 = g^{00}A_0 = -A_0, \qquad A^i = g^{ij}A_j = a^{-2}\delta^{ij}A_j.$$

This makes the scale-factor dependence of spatial components very explicit.

Problem 2. FRW Christoffels

Question. For the flat FRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j,$$

derive $\Gamma^0{}_{ij} = a\dot{a}\,\delta_{ij}$ and $\Gamma^i{}_{0j} = H\delta^i{}_j$.

Solution. The nonzero metric components are

$$g_{00} = -1, \qquad g_{0i} = 0, \qquad g_{ij} = a^2(t)\delta_{ij},$$

with inverse metric

$$g^{00} = -1, \qquad g^{0i} = 0, \qquad g^{ij} = a^{-2}(t)\delta^{ij}.$$

The Christoffel symbols are

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right).$$

Because the metric depends only on $t = x^0$, all spatial derivatives of $g_{\mu\nu}$ vanish. Also,

$$\partial_0 g_{ij} = \frac{\partial}{\partial t}(a^2\delta_{ij}) = 2a\dot{a}\,\delta_{ij}.$$

Derivation of $\Gamma^0{}_{ij}$

Set $\mu = 0$, $\alpha = i$, $\beta = j$:

$$\Gamma^0{}_{ij} = \frac{1}{2}g^{0\nu}\left(\partial_i g_{j\nu} + \partial_j g_{i\nu} - \partial_\nu g_{ij}\right).$$

Since $g^{0\nu}$ is nonzero only for $\nu = 0$,

$$\Gamma^0{}_{ij} = \frac{1}{2}g^{00}\left(\partial_i g_{j0} + \partial_j g_{i0} - \partial_0 g_{ij}\right).$$

Now $g_{i0} = 0$ and $g^{00} = -1$, so

$$\Gamma^0{}_{ij} = \frac{1}{2}(-1)(-\partial_0 g_{ij}) = \frac{1}{2}\partial_0 g_{ij} = \frac{1}{2}(2a\dot{a}\,\delta_{ij}).$$

Therefore,

$$\boxed{\Gamma^0{}_{ij} = a\dot{a}\,\delta_{ij}.}$$

Derivation of $\Gamma^i{}_{0j}$

Now set $\mu = i$, $\alpha = 0$, $\beta = j$:

$$\Gamma^i{}_{0j} = \frac{1}{2} g^{i\nu} \left(\partial_0 g_{j\nu} + \partial_j g_{0\nu} - \partial_\nu g_{0j} \right).$$

Because $g_{0\nu} = 0$ for $\nu \neq 0$ and g_{00} is constant, the last two terms vanish, leaving

$$\Gamma^i{}_{0j} = \frac{1}{2} g^{ik} \partial_0 g_{jk}$$

(where k is a spatial index). Hence

$$\Gamma^i{}_{0j} = \frac{1}{2} \left(a^{-2} \delta^{ik} \right) (2a\dot{a}\, \delta_{jk}) = \frac{\dot{a}}{a} \delta^{ik} \delta_{jk}.$$

Since $\delta^{ik}\delta_{jk} = \delta^i{}_j$,

$$\boxed{\Gamma^i{}_{0j} = \frac{\dot{a}}{a} \delta^i{}_j = H \delta^i{}_j.}$$

By symmetry of the lower indices,

$$\Gamma^i{}_{j0} = \Gamma^i{}_{0j} = H \delta^i{}_j.$$

Problem 3. Continuity equation

Question. Starting from $\nabla_\mu T^{\mu\nu} = 0$ for a perfect fluid in FRW, derive

$$\dot{\rho} + 3H(\rho + P) = 0.$$

Then recover the scalings of matter, radiation, and vacuum energy.

Solution. For a perfect fluid,

$$T^{\mu\nu} = (\rho + P) u^\mu u^\nu + P g^{\mu\nu}.$$

In comoving FRW coordinates the fluid is at rest, so

$$u^\mu = (1, 0, 0, 0), \qquad u_\mu = (-1, 0, 0, 0).$$

Therefore the nonzero components of $T^{\mu\nu}$ are

$$T^{00} = \rho, \qquad T^{0i} = 0, \qquad T^{ij} = P g^{ij} = P a^{-2} \delta^{ij}.$$

To obtain the energy-conservation equation, use the $\nu = 0$ component of

$$\nabla_\mu T^{\mu\nu} = 0.$$

For a rank-2 contravariant tensor,

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu{}_{\mu\alpha} T^{\alpha\nu} + \Gamma^\nu{}_{\mu\alpha} T^{\mu\alpha}.$$

Setting $\nu = 0$ gives

$$0 = \partial_\mu T^{\mu 0} + \Gamma^\mu{}_{\mu\alpha} T^{\alpha 0} + \Gamma^0{}_{\mu\alpha} T^{\mu\alpha}.$$

We evaluate the terms one by one.

First term

Since $T^{i0} = 0$ and $T^{00} = \rho$,

$$\partial_\mu T^{\mu 0} = \partial_0 T^{00} = \dot{\rho}.$$

Second term

Only $\alpha = 0$ contributes because $T^{\alpha 0} = 0$ unless $\alpha = 0$:

$$\Gamma^\mu{}_{\mu\alpha} T^{\alpha 0} = \Gamma^\mu{}_{\mu 0} T^{00}.$$

Now

$$\Gamma^\mu{}_{\mu 0} = \Gamma^0{}_{00} + \Gamma^i{}_{i0} = 0 + 3H = 3H,$$

so this term is

$$3H\rho.$$

Third term

The only nonzero contribution comes from spatial indices:

$$\Gamma^0{}_{\mu\alpha} T^{\mu\alpha} = \Gamma^0{}_{ij} T^{ij}.$$

Using $\Gamma^0{}_{ij} = a\dot{a}\,\delta_{ij}$ and $T^{ij} = Pa^{-2}\delta^{ij}$,

$$\Gamma^0{}_{ij} T^{ij} = (a\dot{a}\,\delta_{ij})(Pa^{-2}\delta^{ij}) = \frac{\dot{a}}{a} P\,\delta_{ij}\delta^{ij} = 3HP.$$

Putting everything together,

$$0 = \dot{\rho} + 3H\rho + 3HP,$$

so

$$\boxed{\dot{\rho} + 3H(\rho + P) = 0.}$$

Scaling with the scale factor

If $P = w\rho$ with constant w, then

$$\dot{\rho} + 3H\rho(1 + w) = 0.$$

Using $H = \dot{a}/a$ and writing $\dot{\rho} = (\mathrm{d}\rho/\mathrm{d}a)\dot{a}$, we get

$$\frac{\mathrm{d}\rho}{\mathrm{d}a}\dot{a} + 3\frac{\dot{a}}{a}\rho(1 + w) = 0.$$

Assuming $\dot{a} \neq 0$,

$$\frac{\mathrm{d}\rho}{\mathrm{d}a} + \frac{3(1 + w)}{a}\rho = 0.$$

This separates as

$$\frac{\mathrm{d}\rho}{\rho} = -3(1 + w)\frac{\mathrm{d}a}{a}.$$

Integrating,

$$\ln\rho = -3(1 + w)\ln a + \text{const},$$

so

$$\boxed{\rho(a) \propto a^{-3(1+w)}.}$$

Hence:

- **Matter (dust):** $w = 0 \Rightarrow \rho_{\rm m} \propto a^{-3}$;
- **Radiation:** $w = 1/3 \Rightarrow \rho_{\rm r} \propto a^{-4}$;
- **Vacuum energy:** $w = -1 \Rightarrow \rho_\Lambda = \text{constant}$.

The extra factor of a^{-1} for radiation comes from the redshifting of each photon energy in addition to the usual dilution of number density.

Problem 4. Acceleration condition

Question. Suppose the Universe contains a single fluid with constant equation of state w. Show that the expansion accelerates only when $w < -1/3$. Give one physical example and one non-example.

Solution. For a homogeneous FRW Universe, the acceleration equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P).$$

If the Universe contains a single fluid with constant equation of state

$$P = w\rho,$$

then

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho(1 + 3w).$$

Assuming a physically sensible positive energy density $\rho > 0$, the sign of $\ddot{a}$ is determined entirely by $1 + 3w$:

- if $1 + 3w > 0$, then $\ddot{a} < 0$ and the expansion decelerates;
- if $1 + 3w < 0$, then $\ddot{a} > 0$ and the expansion accelerates.

Therefore accelerated expansion requires

$$1 + 3w < 0 \qquad \Longrightarrow \qquad \boxed{w < -\frac{1}{3}.}$$

Example. A cosmological constant has

$$w = -1,$$

which satisfies $w < -1/3$, so it drives accelerated expansion.

Non-example. Pressureless matter (dust) has

$$w = 0,$$

so $1 + 3w = 1 > 0$. Therefore it does *not* produce accelerated expansion; instead it causes deceleration. (Another non-example is radiation, with $w = 1/3$.)