Lecture 3: Fundamentals of General Relativity (I)

Gong-Bo Zhao

Abstract

This lecture introduces the pieces of general relativity that are needed in cosmology: tensors, covariant derivatives, Christoffel symbols, geodesics, curvature, the Einstein equation, and the energy-momentum tensor of a perfect fluid. The goal is not a complete GR course, but a cosmology-focused roadmap that makes the Friedmann equations feel motivated rather than mysterious.

Learning goals

After this lecture, students should be able to:

- explain how indices are raised and lowered with the metric;

- write the geodesic equation and interpret Christoffel symbols;

- state the Einstein equation and identify its geometric and matter parts;

- derive the continuity equation for a homogeneous cosmic fluid;

- show how matter, radiation, and vacuum energy scale with the scale factor.

1 Tensors and the metric

In curved spacetime we describe physics using tensors. A contravariant vector is written as

$$A^\mu = (A^0, A^1, A^2, A^3),$$

and the metric $g_{\mu\nu}$ allows us to lower an index:

$$A_\mu = g_{\mu\nu} A^\nu.$$

Similarly,

$$A^\mu = g^{\mu\nu} A_\nu, \qquad g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu.$$

For any 4-vector P^μ, its invariant norm is

$$P_\mu P^\mu = P^2 = g_{\mu\nu} P^\mu P^\nu.$$

The metric can also be used to raise its own indices, for example,

$$g^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} g_{\alpha\beta}.$$

For cosmology we often adopt the metric signature $(-, +, +, +)$ and set $c = 1$ unless otherwise stated. The spacetime interval is

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu.$$

Important reminder. The metric contains all information about distances, angles, and causal structure. In cosmology the metric is not fixed in time: the scale factor $a(t)$ evolves and changes the geometry of spacetime. For the FRW metric, the spatial i-i component of $g_{\mu\nu}$ scales as a^2, while the corresponding component of $g^{\mu\nu}$ scales as a^{-2}.

2 Covariant derivatives and Christoffel symbols

Ordinary partial derivatives do not transform as tensors in curved spacetime, so we introduce the covariant derivative. For a vector,

$$\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\ \nu\alpha} A^\alpha.$$

The quantities $\Gamma^\mu_{\ \alpha\beta}$ are the Christoffel symbols:

$$\Gamma^\mu_{\ \alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta} \right).$$

Christoffel symbols tell us how basis vectors change from point to point. They are not themselves tensors.

Subtle but important point. A common beginner statement is that "the Christoffel symbols vanish if space is not curved." This is too strong. They can vanish at a point in a locally inertial coordinate system, and they vanish everywhere in Cartesian coordinates in flat Minkowski space, but in curvilinear coordinates they can be nonzero even in flat space.

3 Geodesics

Free particles move along geodesics:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\ \alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0,$$

where λ is an affine parameter.

For the spatially flat FRW metric,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j,$$

one may first note that

$$\Gamma^0_{\ \alpha\beta} = \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial t}.$$

Therefore,

$$\Gamma^0_{\ 00} = \Gamma^0_{\ 0i} = \Gamma^0_{\ i0} = 0, \qquad \Gamma^0_{\ ij} = a\dot{a}\,\delta_{ij}, \qquad \Gamma^i_{\ 0j} = \Gamma^i_{\ j0} = H\,\delta^i_{\ j},$$

with $H = \dot{a}/a$.

4 Curvature and the Einstein equation

From the Christoffel symbols we construct curvature tensors, then the Ricci tensor $R_{\mu\nu}$ and Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$. Explicitly,

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\ \mu\nu} - \partial_\nu \Gamma^\alpha_{\ \mu\alpha} + \Gamma^\alpha_{\ \beta\alpha}\Gamma^\beta_{\ \mu\nu} - \Gamma^\alpha_{\ \beta\nu}\Gamma^\beta_{\ \mu\alpha}.$$

The Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R.$$

For the flat FRW background, the curvature quantities are

$$R_{00} = -3\frac{\ddot{a}}{a}, \qquad R_{ij} = \left(2\dot{a}^2 + a\ddot{a}\right)\delta_{ij}, \qquad R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right].$$

The Einstein equation is

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}.$$

Its meaning is simple but profound: spacetime curvature is sourced by energy and momentum.

5   Perfect fluids in cosmology

On large scales, cosmological matter is modeled as a perfect fluid:

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu},$$

where ρ is the energy density, P is the pressure, and u^μ is the fluid 4-velocity.

For a homogeneous and isotropic background, inserting the FRW metric and perfect-fluid stress tensor into the Einstein equation yields the Friedmann equations:

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho,$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P).$$

These equations already contain a major conceptual result: accelerated expansion requires

$$\rho + 3P < 0.$$

For a single fluid with $P = w\rho$, this means

$$w < -\frac{1}{3}.$$

6   Covariant conservation of energy-momentum

General relativity implies local conservation of energy-momentum:

$$\nabla_\mu T^{\mu\nu} = 0.$$

In components, one may write

$$\nabla_\mu T^\mu_{\ \nu} = \frac{\partial T^\mu_{\ \nu}}{\partial x^\mu} + \Gamma^\mu_{\ \alpha\mu} T^\alpha_{\ \nu} - \Gamma^\alpha_{\ \nu\mu} T^\mu_{\ \alpha}.$$

For a homogeneous FRW background this gives the continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0.$$

If we define the equation-of-state parameter $w = P/\rho$, then for constant w,

$$\dot{\rho} + 3H\rho(1 + w) = 0 \qquad \Longrightarrow \qquad \rho(a) \propto a^{-3(1+w)}.$$

Three important special cases are:

$$\rho_{\rm m} \propto a^{-3}, \qquad \rho_{\rm r} \propto a^{-4}, \qquad \rho_\Lambda = \text{constant}.$$

The extra power of a^{-1} for radiation comes from redshifting of photon energies in addition to dilution of number density.

7 The Friedmann equation in multi-component form

In a Universe containing matter, radiation, curvature, and a dark-energy component X,

$$H^2(a) = H_0^2 \left[\Omega_{\rm m} a^{-3} + \Omega_{\rm r} a^{-4} + \Omega_k a^{-2} + \Omega_X X(a) \right],$$

with

$$X(a) = \exp \left[-3 \int_1^a \frac{1 + w(a')}{a'} \, {\rm d}a' \right].$$

and the density parameters satisfy

$$\Omega_{\rm m} + \Omega_{\rm r} + \Omega_k + \Omega_X = 1.$$

For constant w, this reduces to

$$X(a) = a^{-3(1+w)}.$$

8 Summary

This lecture introduced the GR tools needed for cosmology. The main chain of logic is:

1. the metric defines geometry;

2. the Christoffel symbols define covariant differentiation and geodesics;

3. curvature tensors are built from the Christoffel symbols;

4. the Einstein equation links curvature to the stress-energy tensor;

5. for a perfect fluid in FRW, this yields the Friedmann equations and the continuity equation.

Suggested reading

- Dodelson and Schmidt, *Modern Cosmology*, Chapter 2.

- Peacock, *Cosmological Physics*, introductory GR sections.

Homework

1. **Index gymnastics.** Starting from $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_{\ \nu}$, show explicitly how one raises and lowers the indices of a vector and of the metric itself.

2. **FRW Christoffels.** For the flat FRW metric

$$ {\rm d}s^2 = -{\rm d}t^2 + a^2(t)\delta_{ij}{\rm d}x^i {\rm d}x^j,$$

   derive $\Gamma^0_{\ ij} = a\dot{a}\,\delta_{ij}$ and $\Gamma^i_{\ 0j} = H\delta^i_{\ j}$.

3. **Continuity equation.** Starting from $\nabla_\mu T^{\mu\nu} = 0$ for a perfect fluid in FRW, derive

$$\dot{\rho} + 3H(\rho + P) = 0.$$

   Then recover the scalings of matter, radiation, and vacuum energy.

4. **Acceleration condition.** Suppose the Universe contains a single fluid with constant equation of state w. Show that the expansion accelerates only when $w < -1/3$. Give one physical example and one non-example.