# LECTURE 2: FUNDAMENTAL PRINCIPLES AND CONCEPTS IN COSMOLOGY

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## 1. Key concepts in cosmology

1.1. **The scale factor.** The scale factor $a$: it quantifies the 'scale' of the Universe, which is usually normalised to unity at today, *i.e.*, $a(\text{today}) = 1$, and $a(\text{Big Bang}) = 0$.

1.2. **The redshift.** The redshift $z$ measures the relative change in the wavelength between the emitted and observed sources, due to the Doppler effect. Mathematically,

$$z \equiv \frac{\lambda_{\rm obs}}{\lambda_{\rm emt}} - 1 = \frac{a_{\rm obs}}{a_{\rm emt}} - 1 = \frac{1}{a} - 1, \tag{1}$$

where in the last step we have assumed $a_{\rm obs} = 1$ and omit the subscript $_{\rm emt}$.

In cosmology, it is more convenient to use $z$ or $a$ to denote time $t$, *e.g.*, the Big Bang happened at $z \to \infty$, the CMB photons were released at $z \sim 1100$, galaxies formed at $z \lesssim 3$, dark energy starts to dominate the Universe at $z \lesssim 1$. Today, $z = 0$, by definition.

1.3. **The conformal time.**

$$\tau = t/a. \tag{2}$$

1.4. **The Hubble function.**

$$H(a) = \frac{\dot{a}}{a}, \tag{3}$$

where $\dot{} = \frac{\rm d}{{\rm d}t}$. So

$${\rm d}t = \frac{{\rm d}a}{aH}. \tag{4}$$

1.5. **The metric.**

$${\rm d}s^2 = g_{\mu\nu}{\rm d}x^\mu {\rm d}x^\nu = {\rm d}t^2 - a^2({\rm d}x^2 + {\rm d}y^2 + {\rm d}z^2), \tag{5}$$

Isotropy: $a = a(t)$;
Homogeneity: $g_{\mu\nu}(x^\alpha) = g_{\mu\nu}$.

- 2D Euclidian:

$$\begin{aligned} x &= r\ \cos\theta; \\ y &= r\ \sin\theta; \end{aligned} \tag{6}$$

$$\mathrm{d}\ell^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \tag{7}$$

$$g_{1,1} = 1;\ g_{2,2} = r^2 \tag{8}$$

- 4D Minkowski:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{9}$$

$$g_{1,1} = 1;\ g_{2,2} = g_{3,3} = g_{4,4} = -1 \tag{10}$$

- 2D sphere:

$$x^2 + y^2 + z^2 = a^2 \tag{11}$$

$$\begin{aligned} x &= a\ \sin\chi\ \sin\theta; \\ y &= a\ \sin\chi\ \cos\theta; \\ z &= a\ \cos\chi \end{aligned} \tag{12}$$

$$\mathrm{d}\ell^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = a^2(\mathrm{d}\chi^2 + \sin^2\chi\mathrm{d}\theta^2) \tag{13}$$

$$g_{1,1} = a^2;\ g_{2,2} = a^2\sin^2\chi \tag{14}$$

Set $r \equiv \sin\chi$, then

$$\begin{aligned} x &= a\ r\ \sin\theta; \\ y &= a\ r\ \cos\theta; \\ z &= a\ \sqrt{1-r^2} \end{aligned} \tag{15}$$

$$\mathrm{d}\ell^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = a^2\left(\frac{\mathrm{d}r^2}{1-r^2} + r^2\mathrm{d}\theta^2\right) \tag{16}$$

$$g_{1,1} = \frac{a^2}{1-r^2};\ g_{2,2} = a^2r^2 \tag{17}$$

The 2D sphere is a Riemann space with a constant, positive curvature!

- 2D saddle:

$$x^2 + y^2 - z^2 = -a^2 \tag{18}$$

$$\begin{aligned} x &= a\ \sinh\chi\ \sin\theta; \\ y &= a\ \sinh\chi\ \cos\theta; \\ z &= a\ \cosh\chi \end{aligned} \tag{19}$$

Set $r \equiv \sinh\chi$, then

$$\begin{aligned} x &= a\ r\ \sin\theta; \\ y &= a\ r\ \cos\theta; \\ z &= a\ \sqrt{1 - r^2} \end{aligned} \tag{20}$$

$$\mathrm{d}\ell^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2 = a^2 \left( \frac{\mathrm{d}r^2}{1 + r^2} + r^2 \mathrm{d}\theta^2 \right) \tag{21}$$

$$g_{1,1} = \frac{a^2}{1 + r^2};\ g_{2,2} = a^2 r^2 \tag{22}$$

The 2D saddle is a Riemann space with a constant, negative curvature!

- The FRW metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2 \left[ \frac{\mathrm{d}r^2}{1 - Kr^2} + r^2 \left( \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \right) \right] \tag{23}$$

where $K = 0, -1, +1$.

### 1.6. The *confusing* cosmological distances.

- **Proper distance**

  Proper distance roughly corresponds to where a distant object would be at a specific moment of cosmological time, which can change over time due to the expansion of the universe.

$$\ell = \int \mathrm{d}\ell = a \int_0^r \frac{\mathrm{d}r'}{\sqrt{1 - Kr'^2}} = a\chi \tag{24}$$

$$\chi = \begin{array}{ll} \sin^{-1} r & (K = +1) \\ r & (K = 0) \\ \sinh^{-1} r & (K = -1) \end{array} \tag{25}$$

- **Comoving horizon**

$$\eta = \int_0^t \frac{\mathrm{d}t'}{a(t')} = \int_0^a \frac{\mathrm{d}a'}{a'^2 H(a')} \tag{26}$$

- **Comoving distance**: It is the distance as seen from the comoving frame! The comoving distance between observers, *i.e.*, observers that are both moving with the Hubble flow, does not change with time, as comoving distance accounts for the expansion of the universe. Comoving distance is obtained by integrating the proper distances of nearby fundamental observers along the line of sight (LOS), where the proper distance is what a measurement at constant cosmic time would yield;

$$\chi = \int_t^{t_0} \frac{\mathrm{d}t'}{a(t')} = \int_a^1 \frac{\mathrm{d}a'}{a'^2 H(a')} \tag{27}$$

- **Angular diameter distance**: An object of size $\ell$ at redshift $z$ that appears to have angular size $\theta$. So

$$\theta = \frac{\ell/a}{\chi}. \tag{28}$$

One the other hand, if we define an "angular diameter distance" $d_A(z)$ so that $\theta = \ell/d_A$, then

$$d_A = a\chi = \frac{\chi}{1+z}. \tag{29}$$

- **Luminosity distance**: If the intrinsic luminosity $L$ of a distant object is known, we can calculate its luminosity distance by measuring the flux $S$ and determine $d_L(z) = \sqrt{L/4\pi S}$. This quantity is important for measurements of standard candles like type Ia supernovae, which were first used to discover the acceleration of the expansion of the universe.

$$d_L = \chi/a. \tag{30}$$

## 2. Fundamental principles in cosmology

### 2.1. Energy conservation.

**2.2. Homogeneity and Isotropy.** On large scales, the distribution of matter, and the spacetime, is homogeneous and isotropic.

**Homogeneity**: All places look alike. No special location.
**Isotropy**: All directions look alike. No special direction.

**2.3. The equivalence principle.** The gravitational and inertial mass are identical.

We can test the equivalence principle using observations, as we can measure the lensing (gravitational) mass, and the inertial (dynamical) mass, respectively. See [3].

**2.4. Copernican principle.** We are not in a special place in the Universe. See tests in [4, 5, 6, 7, 8]

2.5. **The CPT symmetry.** The symmetry of charge-parity-time reversal. See a test in [9, 10].

### Homework

Read Chapter 2.2 of [1], Chapter 3 of [2] (you can ignore the formulae for now), and papers [3, 4, 5, 6, 7, 8, 9, 10]. Try to get the big picture behind the maths!

## References

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