Solutions to the Homework in Lecture 2
Fundamental Principles and Concepts in Cosmology

These notes provide worked solutions to the homework problems in Lecture 2. Throughout, we use the convention $a(t_0) = 1$ and $1 + z = 1/a$.

Problem 1. Conceptual check

Question. Explain in your own words why cosmological redshift is better interpreted as wavelength stretching in an expanding spacetime rather than as an ordinary special-relativistic Doppler effect.

Solution. In special relativity, a Doppler shift is defined in a fixed, non-expanding spacetime: source and observer move through the same static background, and the wavelength change is caused by their relative velocity. Cosmological redshift is different. On large scales, galaxies that follow the Hubble flow are not best described as moving through a rigid pre-existing space; instead, the spatial part of the metric itself changes with time through the scale factor $a(t)$. As the Universe expands, the wavelength of a freely propagating photon stretches in proportion to the scale factor,

$$\lambda \propto a(t),$$

so that

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = \frac{a(t_0)}{a(t_{\text{em}})}.$$

This relation follows directly from the FLRW geometry and does not require interpreting the effect as a sequence of ordinary Doppler kicks.

A useful way to say this is:

- for *nearby* galaxies, where spacetime curvature and the evolution of $a(t)$ over the light-travel time are small, the redshift can be approximated by an ordinary Doppler shift;
- for *cosmological* distances, the cleaner and more general interpretation is that photon wavelengths are stretched by the expansion of spacetime itself.

So the reason cosmologists prefer the "stretching of spacetime" language is that it is the interpretation naturally built into the FLRW metric and it remains valid on the very large scales where the special-relativistic Doppler picture becomes incomplete.

Problem 2. Derivation of conformal time

Question. Starting from $\mathrm{d}\tau = \mathrm{d}t/a$ and $\mathrm{d}t = \mathrm{d}a/(aH)$, show that

$$\tau(a) = \int_0^a \frac{\mathrm{d}a'}{a'^2 H(a')}.$$

Evaluate $\tau(a)$ analytically for a flat matter-only Universe with $H(a) = H_0 a^{-3/2}$.

Solution. Start from the definition of conformal time,

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{a}.$$

Using

$$\mathrm{d}t = \frac{\mathrm{d}a}{aH(a)},$$

we immediately obtain

$$\mathrm{d}\tau = \frac{1}{a}\frac{\mathrm{d}a}{aH(a)} = \frac{\mathrm{d}a}{a^2H(a)}.$$

Integrating from the Big Bang ($a' = 0$) to some scale factor a gives

$$\tau(a) = \int_0^a \frac{\mathrm{d}a'}{a'^2H(a')}.$$

This is the required result.

Now consider a flat matter-only Universe, for which

$$H(a) = H_0a^{-3/2}.$$

Substituting into the integral,

$$\tau(a) = \int_0^a \frac{\mathrm{d}a'}{a'^2H_0a'^{-3/2}} = \frac{1}{H_0}\int_0^a a'^{-1/2}\,\mathrm{d}a'.$$

Since

$$\int a'^{-1/2}\,\mathrm{d}a' = 2a'^{1/2},$$

we get

$$\tau(a) = \frac{2}{H_0}\sqrt{a}.$$

Therefore, for a flat matter-only Universe,

$$\boxed{\tau(a) = \frac{2}{H_0}\,a^{1/2}}.$$

Because $a = (1+z)^{-1}$, this can also be written as

$$\tau(z) = \frac{2}{H_0\sqrt{1+z}}.$$

Problem 3. Distance duality

Question. Starting from the definitions of flux and angular size, derive

$$D_L = (1+z)^2D_A.$$

State clearly which physical assumptions are used.

Solution. Let a source have intrinsic luminosity L and let its observed flux be F. By definition,

$$F = \frac{L}{4\pi D_L^2}.$$

We now relate D_L to the transverse comoving distance D_M.

Suppose the source emits photons isotropically. By the time those photons reach us, they are spread over an area $4\pi D_M^2$ in comoving terms, but two additional redshift effects reduce the observed flux:

1. each photon energy is redshifted by a factor $(1+z)^{-1}$;
2. the photon arrival rate is time-dilated by another factor $(1+z)^{-1}$.

Hence

$$F = \frac{L}{4\pi D_M^2 (1+z)^2}.$$

Comparing with the definition of D_L,

$$\frac{L}{4\pi D_L^2} = \frac{L}{4\pi D_M^2 (1+z)^2},$$

so

$$D_L = (1+z) D_M.$$

Now consider angular size. If an object has physical transverse size ℓ at emission and subtends an angle θ, then by definition

$$D_A = \frac{\ell}{\theta}.$$

In FLRW geometry, the physical transverse size at emission is related to the transverse comoving distance by

$$\ell = a_{\rm em} D_M \theta = \frac{D_M}{1+z}\theta,$$

because $a_{\rm em} = 1/(1+z)$. Therefore

$$D_A = \frac{\ell}{\theta} = \frac{D_M}{1+z}.$$

Rearranging,

$$D_M = (1+z) D_A.$$

Substituting into $D_L = (1+z) D_M$ gives

$$D_L = (1+z)^2 D_A.$$

Thus,

$$\boxed{D_L = (1+z)^2 D_A.}$$

Physical assumptions. The derivation uses the following assumptions:

- spacetime is described by a metric theory so that light travels on null geodesics;
- photon number is conserved between source and observer (no absorption, scattering, or photon conversion);
- the source and observer are connected by the usual FLRW geometric relation between transverse comoving distance and physical size.

These are the assumptions behind the Etherington distance-duality relation.

Problem 4. Worked calculation in Einstein–de Sitter

Question. In an Einstein–de Sitter Universe ($\Omega_{\rm m} = 1$, $\Omega_\Lambda = \Omega_k = 0$), show that

$$\chi(z) = \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right).$$

Then compute D_A and D_L at $z = 1$ for $H_0 = 70\ {\rm km\,s^{-1}\,Mpc^{-1}}$.

Solution. In an Einstein–de Sitter Universe,

$$H(z) = H_0(1+z)^{3/2}.$$

The comoving radial distance is

$$\chi(z) = c\int_0^z \frac{{\rm d}z'}{H(z')} = \frac{c}{H_0}\int_0^z (1+z')^{-3/2}\,{\rm d}z'.$$

Let $u = 1 + z'$, so ${\rm d}u = {\rm d}z'$. Then

$$\chi(z) = \frac{c}{H_0}\int_1^{1+z} u^{-3/2}\,{\rm d}u.$$

Since

$$\int u^{-3/2}\,{\rm d}u = -2u^{-1/2},$$

we obtain

$$\chi(z) = \frac{c}{H_0}\left[-2u^{-1/2}\right]_1^{1+z} = \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right).$$

Therefore,

$$\boxed{\chi(z) = \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right).}$$

Because the Universe is spatially flat, the transverse comoving distance is simply

$$D_M = \chi.$$

Hence

$$D_A = \frac{\chi}{1+z}, \qquad D_L = (1+z)\chi.$$

At $z = 1$,

$$\chi(1) = \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{2}}\right).$$

Using $c = 299792.458\ {\rm km\,s^{-1}}$ and $H_0 = 70\ {\rm km\,s^{-1}\,Mpc^{-1}}$,

$$\frac{c}{H_0} = 4282.75\ {\rm Mpc}.$$

Therefore

$$\chi(1) = 2(4282.75\ {\rm Mpc})\left(1 - \frac{1}{\sqrt{2}}\right) \approx 2508.8\ {\rm Mpc}.$$

So

$$D_A(z=1) = \frac{\chi(1)}{2} \approx 1254.4\ {\rm Mpc} \approx 1.25\ {\rm Gpc},$$

and

$$D_L(z=1) = 2\chi(1) \approx 5017.6\ {\rm Mpc} \approx 5.02\ {\rm Gpc}.$$

Thus,

$$\boxed{D_A(z=1) \approx 1.25\ {\rm Gpc}, \qquad D_L(z=1) \approx 5.02\ {\rm Gpc}.}$$

As a quick check,

$$D_L = (1+z)^2 D_A = 4D_A,$$

and indeed 4 × 1.25 Gpc ≈ 5.0 Gpc.