Lecture 2: Fundamental Principles and Concepts in Cosmology

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Abstract

This lecture introduces the basic language of cosmology: scale factor, redshift, Hubble expansion, conformal time, the FLRW metric, and the main cosmological distance measures. We also summarize the physical principles that underlie the standard cosmological model. The emphasis is on clear definitions and on avoiding several common confusions, especially those involving conformal time and distance measures.

Learning goals

After this lecture, students should be able to:

- explain the meanings of the scale factor, redshift, and Hubble parameter;
- distinguish cosmic time, conformal time, and horizon distance;
- write the FLRW metric and interpret the curvature parameter;
- define proper, comoving, angular-diameter, and luminosity distances;
- state the cosmological principle, the Copernican principle, and the equivalence principle.

1 Scale factor and redshift

The scale factor $a(t)$ describes how all large cosmological distances change with time. By convention we set

$$a(t_0) = 1$$

today. The redshift z of a photon emitted at time $t_{\rm em}$ and observed today is

$$1 + z = \frac{\lambda_{\rm obs}}{\lambda_{\rm em}} = \frac{a(t_0)}{a(t_{\rm em})} = \frac{1}{a(t_{\rm em})}.$$

A useful conceptual point is that a galaxy redshift in cosmology is not merely an ordinary Doppler shift from motion through static space. On very large scales, it is better understood as the stretching of photon wavelengths by the expansion of spacetime itself.

Because a and z are monotonic functions of time in the expanding Universe, cosmologists often use them as alternative time variables. For example:

- today: $a = 1$, $z = 0$;
- recombination: $z \approx 1100$;
- matter-radiation equality: $z \approx 3400$;
- the early Universe: $z \gg 1$.

2 The Hubble parameter

The Hubble parameter is defined by

$$H(t) \equiv \frac{\dot{a}}{a},$$

where a dot denotes differentiation with respect to cosmic time t. It measures the fractional expansion rate of the Universe.

A useful differential relation is

$$\mathrm{d}t = \frac{\mathrm{d}a}{aH(a)}.$$

This identity lets us convert time integrals into integrals over a or z.

At low redshift, the Hubble parameter today is written as

$$H_0 = 100\,h\ \mathrm{km\,s^{-1}\,Mpc^{-1}}.$$

3 Conformal time

A common source of confusion is the definition of conformal time. It is *not* generally true that $\tau = t/a$. Instead, conformal time is defined by

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{a(t)}.$$

Therefore

$$\tau(t) = \int^{t} \frac{\mathrm{d}t'}{a(t')},$$

up to an arbitrary additive constant.

Why is conformal time useful? In the FLRW metric it pulls out the overall expansion factor and makes light propagation look more Minkowski-like. In particular, for a radial light ray in a flat Universe, the null condition implies

$$\mathrm{d}\chi = \mathrm{d}\tau,$$

where χ is comoving radial distance. This is why conformal time is closely related to particle horizons and to CMB calculations.

Two related but different quantities.

- The **conformal time** at epoch t is $\tau(t) = \int^t \mathrm{d}t'/a(t')$.
- The **comoving particle horizon** is the comoving distance that light could have traveled since the beginning of the expansion:

$$\chi_{\mathrm{hor}}(t) = \int_0^{t} \frac{\mathrm{d}t'}{a(t')}.$$

With a suitable choice of origin these can be numerically equal, but conceptually they should not be conflated.

4 The FLRW metric and spatial curvature

The standard metric for a homogeneous and isotropic Universe is the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right].$$

Here k is the normalized spatial-curvature parameter:

$$k = \begin{cases} +1 & \text{closed spatial geometry,} \\ 0 & \text{flat spatial geometry,} \\ -1 & \text{open spatial geometry.} \end{cases}$$

Homogeneity means that no spatial point is special on large scales. Isotropy means that no direction is special. Together, these symmetries strongly constrain the metric to the FLRW form.

5 Cosmological distances

Different observations probe different notions of distance, so it is important to keep them separate.

Comoving radial distance

For a source at redshift z,

$$\chi(z) = c\int_0^z \frac{\mathrm{d}z'}{H(z')}.$$

This quantity factors out the expansion and is the most convenient radial distance in theory.

Proper distance

At a fixed cosmic time, the proper radial distance is

$$D_{\mathrm{prop}}(t) = a(t)\,\chi.$$

Because $a(t)$ changes with time, proper distance between distant galaxies changes even if their comoving distance stays fixed.

Transverse comoving distance

Curvature modifies transverse separations. Define

$$D_M(z) = \begin{cases} \frac{c}{H_0\sqrt{\Omega_k}}\sinh\!\left(\sqrt{\Omega_k}\,H_0\chi/c\right), & \Omega_k > 0, \\ \chi, & \Omega_k = 0, \\ \frac{c}{H_0\sqrt{-\Omega_k}}\sin\!\left(\sqrt{-\Omega_k}\,H_0\chi/c\right), & \Omega_k < 0. \end{cases}$$

Angular-diameter distance

If an object with physical transverse size ℓ subtends angle θ, then

$$D_A \equiv \frac{\ell}{\theta} = \frac{D_M}{1+z}.$$

Luminosity distance

If an object of intrinsic luminosity L is observed with flux F, then

$$D_L \equiv \sqrt{\frac{L}{4\pi F}} = (1+z)D_M.$$

Therefore,

$$D_L = (1+z)^2 D_A.$$

This is the Etherington distance-duality relation, a very important consistency relation in cosmology.

6 Useful time and distance relations

It is often helpful to switch among t, a, and z:

$$1+z = \frac{1}{a}, \qquad \mathrm{d}t = \frac{\mathrm{d}a}{aH(a)}, \qquad \mathrm{d}t = -\frac{\mathrm{d}z}{(1+z)H(z)}.$$

For conformal time,

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{a} = \frac{\mathrm{d}a}{a^2 H(a)} = -\frac{\mathrm{d}z}{H(z)}.$$

In units with $c = 1$, the last expression shows immediately why conformal time and comoving distance are so closely linked.

7 Foundational principles

The cosmological principle

On sufficiently large scales, the Universe is homogeneous and isotropic. This is a modeling assumption, but it is strongly supported by observations of the CMB and large-scale galaxy surveys.

The Copernican principle

We do not occupy a special position in the Universe. This principle is weaker than exact homogeneity, but it is a key idea behind modern cosmology.

The equivalence principle

Locally, gravitational and inertial mass are equivalent. This is one of the foundations of general relativity.

About CPT symmetry

Charge-parity-time reversal symmetry is a fundamental symmetry in particle physics and is occasionally tested using cosmological observations. It is interesting, but it is not part of the minimal geometric setup of standard cosmology in the same way that the previous three principles are.

8 Summary

This lecture introduced the main variables and distances used throughout cosmology. The most important takeaways are:

- redshift and scale factor are related by $1 + z = 1/a$;
- conformal time is defined by $\mathrm{d}\tau = \mathrm{d}t/a$, not by $\tau = t/a$;
- the FLRW metric describes a homogeneous and isotropic Universe;
- different observations use different distance measures, linked by

$$D_L = (1 + z)^2 D_A.$$

These definitions will be used repeatedly in later lectures on general relativity, perturbation theory, supernovae, BAO, and the CMB.

Suggested reading

- Dodelson and Schmidt, *Modern Cosmology*, Chapter 2.
- Peacock, *Cosmological Physics*, Chapter 3.

Homework

1. **Conceptual check.** Explain in your own words why cosmological redshift is better interpreted as wavelength stretching in an expanding spacetime rather than as an ordinary special-relativistic Doppler effect.

2. **Derivation.** Starting from $\mathrm{d}\tau = \mathrm{d}t/a$ and $\mathrm{d}t = \mathrm{d}a/(aH)$, show that

$$\tau(a) = \int_0^a \frac{\mathrm{d}a'}{a'^2 H(a')}.$$

   Evaluate $\tau(a)$ analytically for a flat matter-only Universe with $H(a) = H_0 a^{-3/2}$.

3. **Distance duality.** Starting from the definitions of flux and angular size, derive

$$D_L = (1 + z)^2 D_A.$$

   State clearly which physical assumptions are used.

4. **Worked calculation.** In an Einstein-de Sitter Universe ($\Omega_\mathrm{m} = 1$, $\Omega_\Lambda = \Omega_k = 0$), show that

$$\chi(z) = \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right).$$

   Then compute D_A and D_L at $z = 1$ for $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$.