Solutions to the Homework in Lecture 15
Fundamentals of Observational Cosmology

These notes provide worked solutions to the homework problems in Lecture 15.

Problem 1. Quadratic inflation in slow roll

Question. For $V(\phi) = m^2\phi^2/2$, use the slow-roll relations to show that

$$r \simeq \frac{8}{N_\star}, \qquad n_s \simeq 1 - \frac{2}{N_\star}.$$

Evaluate these at $N_\star = 50$ and 60, and compare qualitatively with the low-r preference discussed in the lecture.

Solution. For the quadratic potential,

$$V(\phi) = \frac{1}{2}m^2\phi^2, \qquad V'(\phi) = m^2\phi, \qquad V''(\phi) = m^2.$$

Using the slow-roll definitions from the lecture notes,

$$\epsilon = \frac{m_{\rm Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2, \qquad \eta = \frac{m_{\rm Pl}^2}{8\pi}\frac{V''}{V},$$

we get

$$\frac{V'}{V} = \frac{2}{\phi}, \qquad \frac{V''}{V} = \frac{2}{\phi^2},$$

so

$$\epsilon = \frac{m_{\rm Pl}^2}{4\pi\phi^2}, \qquad \eta = \frac{m_{\rm Pl}^2}{4\pi\phi^2} = \epsilon.$$

Now use the slow-roll e-fold relation

$$N_\star \simeq \frac{8\pi}{m_{\rm Pl}^2}\int_{\phi_{\rm end}}^{\phi_\star}\frac{V}{V'}\,{\rm d}\phi.$$

For $V/V' = \phi/2$ this becomes

$$N_\star \simeq \frac{8\pi}{m_{\rm Pl}^2}\int_{\phi_{\rm end}}^{\phi_\star}\frac{\phi}{2}\,{\rm d}\phi = \frac{2\pi}{m_{\rm Pl}^2}\left(\phi_\star^2 - \phi_{\rm end}^2\right).$$

Inflation ends when $\epsilon(\phi_{\rm end}) \simeq 1$, hence

$$1 = \frac{m_{\rm Pl}^2}{4\pi\phi_{\rm end}^2} \qquad \Longrightarrow \qquad \phi_{\rm end}^2 = \frac{m_{\rm Pl}^2}{4\pi}.$$

Substituting this into the expression for $N_\star$ gives

$$\phi_\star^2 = \frac{m_{\rm Pl}^2}{4\pi}(2N_\star + 1).$$

Therefore

$$\epsilon_\star = \eta_\star = \frac{m_{\rm Pl}^2}{4\pi\phi_\star^2} = \frac{1}{2N_\star + 1} \simeq \frac{1}{2N_\star}$$

for large $N_\star$.

The tensor-to-scalar ratio is then

$$r = 16\epsilon_\star = \frac{16}{2N_\star + 1} \simeq \frac{8}{N_\star}.$$

The scalar tilt is

$$n_s - 1 = -6\epsilon_\star + 2\eta_\star = -4\epsilon_\star = -\frac{4}{2N_\star + 1} \simeq -\frac{2}{N_\star},$$

so

$$n_s \simeq 1 - \frac{2}{N_\star}.$$

Numerically,

$$N_\star = 50: \qquad r \simeq \frac{8}{50} = 0.16, \qquad n_s \simeq 1 - \frac{2}{50} = 0.96,$$

and

$$N_\star = 60: \qquad r \simeq \frac{8}{60} \approx 0.133, \qquad n_s \simeq 1 - \frac{2}{60} \approx 0.9667.$$

These n_s values are reasonable, but the predicted tensor amplitude is fairly large. Since the lecture emphasized that present data prefer a small tensor-to-scalar ratio, this means that simple quadratic inflation is under more pressure than plateau models such as Starobinsky or more general α-attractors, which predict much smaller r.

Problem 2. Why Lyα helps running more than amplitude

Question. Starting from

$$\ln \mathcal{P}_{\mathcal{R}}(k) = \ln A_s + (n_s - 1)\ln(k/k_*) + \frac{\alpha_s}{2}\ln^2(k/k_*) + \cdots,$$

explain in words why adding DESI Lyα data helps more with α_s than with A_s.

Solution. The key point is that A_s and α_s affect the spectrum in qualitatively different ways.

- $\ln A_s$ mainly shifts the whole spectrum up or down. It is an overall normalization.
- α_s changes the *curvature* of $\ln \mathcal{P}_{\mathcal{R}}(k)$ as a function of $\ln k$. Its contribution is weighted by $\frac{1}{2}\ln^2(k/k_*)$, so it matters most far away from the pivot scale.

A Lyα measurement is especially useful because it extends the observed range to much smaller scales, that is, to much larger k than the CMB pivot scale and also beyond the galaxy-clustering scales that dominate many large-scale-structure analyses. This increases the lever arm in $\ln k$, which is exactly what one needs to measure the $\ln^2(k/k_*)$ term.

A compact way to see the same idea is

$$\frac{\partial \ln \mathcal{P}_{\mathcal{R}}}{\partial \ln A_s} = 1, \qquad \frac{\partial \ln \mathcal{P}_{\mathcal{R}}}{\partial \alpha_s} = \frac{1}{2}\ln^2(k/k_*).$$

The sensitivity to α_s therefore grows the farther one moves from the pivot, while the sensitivity to A_s does not.

In addition, the amplitude is already constrained fairly well by the CMB, and in large-scale-structure data the overall amplitude can be partly degenerate with quantities such as galaxy or forest bias, optical-depth related amplitude information, and late-time growth. So DESI Lyα is especially valuable not because it simply remeasures the amplitude, but because it adds a long lever arm in scale and therefore sharpens constraints on running.

Problem 3. Scale-dependent bias from local PNG

Question. Use the scale-dependent bias formula $\Delta b \propto f_{\rm NL}^{\rm loc}/k^2$ to answer the following. If one compares two modes with $k_2 = 2k_1$, by what factor does the PNG signal change? Why are high-bias tracers such as quasars especially valuable?

Solution. Since

$$\Delta b \propto \frac{f_{\rm NL}^{\rm loc}}{k^2},$$

doubling the wavenumber suppresses the signal by a factor of four:

$$\frac{\Delta b(k_2)}{\Delta b(k_1)} = \frac{k_1^2}{k_2^2} = \frac{k_1^2}{(2k_1)^2} = \frac{1}{4}.$$

So the PNG signal at $k_2 = 2k_1$ is only one quarter as large as at k_1.

This immediately explains why the largest scales are the most important for local primordial non-Gaussianity: the effect grows like k^{-2} and is therefore strongest at small k.

High-bias tracers such as quasars are especially useful because the scale-dependent PNG correction is larger for more highly biased tracers. In practical formulas one often finds an extra factor roughly proportional to $(b-1)$ multiplying the $f_{\rm NL}^{\rm loc}/k^2$ term. Quasars are attractive because they are highly biased and live at relatively high redshift, so they amplify the signal while also probing very large cosmic volumes.

Problem 4. Which inflation models are favoured or disfavoured?

Question. Current data are consistent with small r, negligible local $f_{\rm NL}$, and no significant running. Name one broad class of inflation models that is favoured by this pattern and one broad class that is disfavoured, and give one sentence of justification for each answer.

Solution. A broad class that is **favoured** is

$$\boxed{\text{plateau-type single-field slow-roll models}}$$

such as Starobinsky inflation or related α-attractors. These typically predict a red scalar tilt close to the observed value, a very small tensor-to-scalar ratio, and no large local non-Gaussianity or running.

A broad class that is **disfavoured** is

$$\boxed{\text{simple large-field monomial models}}$$

such as $\lambda\phi^4$ and, to a lesser extent, the classic quadratic model $m^2\phi^2/2$. These models generically predict a tensor amplitude that is larger than the low-r pattern highlighted in the lecture and are therefore less compatible with present observations than plateau models.