LECTURE 15: INFLATION

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Learning goals

After this lecture, students should be able to:

- explain why inflationary model predictions are best viewed as trajectories or bands in the (n_s, r) plane rather than isolated points;
- derive the leading slow-roll predictions for A_s, n_s, and r, and understand why running and primordial non-Gaussianity are especially informative beyond these lowest-order observables;
- describe how large-scale structure complements the CMB by probing a much larger three-dimensional volume and a much wider range of scales in k;
- explain the logic behind DESI constraints on local primordial non-Gaussianity and on the running of the primordial power spectrum;
- summarize the main lessons from recent Planck and DESI inflation analyses.

1. Motivation for Inflation

The standard (big-bang) cosmology successfully describes the thermal history of the Universe, yet it faces several conceptual challenges:

- **Horizon problem**: the observed CMB is uniform to one part in 10^5 across regions that were causally disconnected at the time of last scattering.
- **Flatness problem**: the present near-critical density ($|\Omega_k| \ll 1$) requires extreme fine-tuning of the initial curvature.
- **Monopole problem**: grand-unified theories generically overproduce magnetic monopoles that are not observed.
- **Origin of perturbations**: the big-bang framework offers no mechanism for the nearly scale-invariant spectrum of primordial fluctuations measured in the CMB.

An early epoch of quasi-exponential expansion—*cosmic inflation*—resolves these problems in a single stroke.

2. Scalar-Field Dynamics

We model inflation with a minimally-coupled scalar field ϕ (the *inflaton*) rolling slowly down its potential $V(\phi)$ in a spatially flat Friedmann–Robertson–Walker (FRW) universe.

$$H^2 = \frac{8\pi G}{3}\Big[\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big], \qquad \dot{H} = -4\pi G\dot{\phi}^2, \tag{1}$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{2}$$

When does inflation actually occur? A useful alternative to the potential slow-roll parameters is the Hubble slow-roll parameter

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2}. \tag{3}$$

Using $\dot{H} = -4\pi G\dot{\phi}^2$, one finds

$$\frac{\ddot{a}}{a} = H^2(1 - \epsilon_H), \qquad \epsilon_H = \frac{4\pi G\dot{\phi}^2}{H^2}. \tag{4}$$

Therefore accelerated expansion requires $\epsilon_H < 1$. During slow roll, $\epsilon_H \simeq \epsilon$, so the familiar condition $\epsilon \ll 1$ has an immediate physical meaning: the inflaton kinetic energy must remain subdominant so that the expansion is quasi-exponential.

Slow-roll approximation. Inflation occurs if the potential energy dominates, $V \gg \dot{\phi}^2$, which can be recast as the *slow-roll* conditions

$$\epsilon \equiv \frac{m_{\rm Pl}^2}{16\pi}\Big(\frac{V'}{V}\Big)^2 \ll 1, \qquad \eta \equiv \frac{m_{\rm Pl}^2}{8\pi}\frac{V''}{V} \ll 1. \tag{5}$$

Then Eqs. (1)–(2) reduce to

$$H^2 \simeq \frac{8\pi G}{3}V, \qquad 3H\dot{\phi} \simeq -V'. \tag{6}$$

The number of *e-folds* generated between ϕ_i and ϕ_f is

$$N \equiv \int_{t_i}^{t_f} H\mathrm{d}t \simeq \frac{8\pi}{m_{\rm Pl}^2}\int_{\phi_f}^{\phi_i}\frac{V}{V'}\mathrm{d}\phi. \tag{7}$$

Successful inflation requires $N \gtrsim 50$–60 to solve the horizon problem for modes observed today.

3. Prototype Models

A vast zoo of potentials satisfy the slow-roll criteria. Popular examples include

(1) **Chaotic inflation**: $V(\phi) = \frac{1}{2}m^2\phi^2$ or $\lambda\phi^4$.
(2) **Starobinsky R^2 inflation**: $V(\phi) = \Lambda^4\big(1 - e^{-\sqrt{2/3}\,\phi/m_{\rm Pl}}\big)^2$.
(3) **Plateau models** (*e.g.* small-field, hilltop, or E-model potentials).
(4) **Hybrid inflation**: inflation ends through a second (waterfall) field.

Each predicts distinct scalar spectral index n_s and tensor-to-scalar ratio r, providing an observational handle to discriminate between them.

Model tracks in the (n_s, r) plane. For many single-field models the observables can be written approximately in terms of the number of e-folds $N_\star$ between horizon exit of the pivot mode and the end of inflation. For a monomial potential $V(\phi) \propto \phi^p$,

$$n_s \simeq 1 - \frac{p+2}{2N_\star}, \qquad r \simeq \frac{4p}{N_\star}. \tag{8}$$

For Starobinsky or more general α-attractor plateau models,

$$n_s \simeq 1 - \frac{2}{N_\star}, \qquad r \simeq \frac{12\alpha}{N_\star^2}, \tag{9}$$

with $\alpha = 1$ reproducing the classic R^2 prediction. For example, at $N_\star = 60$, quadratic inflation gives roughly $(n_s, r) \simeq (0.967, 0.13)$, whereas Starobinsky gives $(0.967, 0.003)$. This is an excellent classroom illustration of why a small measured tensor amplitude strongly favours plateau-type models even when two models predict similar values of n_s.

4. Generation of Perturbations

Quantum fluctuations of ϕ and the metric are stretched outside the horizon during inflation. In the slow-roll limit, the dimensionless power spectra are

$$\mathcal{P}_\mathcal{R}(k) \;=\; A_s \left(\frac{k}{k_*}\right)^{n_s-1},$$

$$n_s - 1 \;\equiv\; \left.\frac{d\ln\mathcal{P}_\mathcal{R}(k)}{d\ln k}\right|_{k=k_*}.$$

$$\mathcal{P}_\mathcal{R}(k) = \left(\frac{H^2}{2\pi\dot\phi}\right)^2_{k=aH} = \frac{1}{12\pi^2}\frac{V}{m_{\rm Pl}^4\epsilon}, \tag{10}$$

$$\mathcal{P}_h(k) = \frac{2}{3\pi^2}\frac{V}{m_{\rm Pl}^4}, \qquad r \equiv \frac{\mathcal{P}_h}{\mathcal{P}_\mathcal{R}} = 16\epsilon. \tag{11}$$

Spectral observables. To first order in slow-roll, the scalar spectral index and its running are

$$n_s - 1 = -6\epsilon + 2\eta, \qquad \frac{\mathrm{d}n_s}{\mathrm{d}\ln k} = 16\epsilon\eta - 24\epsilon^2 - 2\xi^2, \tag{12}$$

with $\xi^2 \equiv (m_{\rm Pl}^4/64\pi^2)(V'V'''/V^2)$.

Beyond a pure power-law primordial spectrum. A more general parametrization of the curvature spectrum is

$$\ln\mathcal{P}_\mathcal{R}(k) = \ln A_s + (n_s-1)\ln\frac{k}{k_*} + \frac{\alpha_s}{2}\ln^2\frac{k}{k_*} + \frac{\beta_s}{6}\ln^3\frac{k}{k_*} + \cdots, \tag{13}$$

where

$$\alpha_s \equiv \frac{dn_s}{d\ln k}, \qquad \beta_s \equiv \frac{d^2 n_s}{d(\ln k)^2}. \tag{14}$$

In smooth slow-roll models one typically expects $|\alpha_s| \sim \mathcal{O}((n_s-1)^2)$ and $|\beta_s|$ even smaller. A significant detection of nonzero running or running-of-running would therefore hint at transient violations of slow roll, features in the potential, multiple fields, or other departures from the simplest attractor picture.

Primordial non-Gaussianity as a discriminator of inflation. The power spectrum is only the beginning. Inflation also predicts higher-order correlation functions. One convenient local ansatz is

$$\Phi(\boldsymbol{x}) = \phi_G(\boldsymbol{x}) + f_{\rm NL}^{\rm loc}\left[\phi_G^2(\boldsymbol{x}) - \langle \phi_G^2 \rangle\right] + \cdots , \tag{15}$$

and more generally the bispectrum is defined by

$$\langle \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle = (2\pi)^3 \delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\, B_\zeta(k_1,k_2,k_3). \tag{16}$$

The *shape* of B_ζ carries physical information: local shapes are typically associated with super-horizon mode coupling or multiple light fields, whereas equilateral and orthogonal shapes often arise from non-canonical dynamics or higher-derivative interactions. In the simplest single-field attractor models, the squeezed limit obeys Maldacena's consistency relation [10],

$$\lim_{k_L \ll k_S} B_\zeta(k_L,k_S,k_S) \simeq -(n_s-1)\, P_\zeta(k_L) P_\zeta(k_S), \tag{17}$$

so the observable local signal is expected to be very small. This is why local primordial non-Gaussianity is such a powerful test: a convincing detection at observable amplitude would point beyond the simplest single-field slow-roll framework.

Why large-scale structure is a complementary inflation probe. DESI does not compete directly with CMB B-mode experiments on the tensor amplitude r, but it probes inflation in two other important ways. First, it extends the lever arm in wavenumber k, which is crucial for testing running of the primordial spectrum. Second, it measures an enormous three-dimensional volume, which is ideal for constraining local primordial non-Gaussianity through the scale-dependent bias of halos and galaxies. A standard approximation is [11]

$$\Delta b(k,z) \simeq \frac{3\Omega_m H_0^2\, \delta_c\, [b_1(z)-p]}{k^2 T(k) D(z)}\, f_{\rm NL}^{\rm loc}, \tag{18}$$

where $\delta_c \simeq 1.686$, $T(k)$ is the transfer function, $D(z)$ is the linear growth factor, and p encodes tracer formation history. The k^{-2} enhancement means that very large scales and highly biased tracers such as quasars carry disproportionate information. This is precisely why DESI has become a serious inflation experiment, not only a dark-energy survey. Earlier large-scale-structure constraints of this general kind were already demonstrated with BOSS galaxy power-spectrum and bispectrum analyses [12, 13].

5. Observational Status (*Planck* 2018 + BK18)

For a student-facing summary, it is helpful to separate the scalar-tilt measurement from the tensor bound. The *Planck* 2018 inflation analysis found $n_s = 0.9649 \pm 0.0042$ at 68% confidence, with no evidence for a scale dependence of the tilt [6]. A more up-to-date tensor bound comes from the BICEP/Keck BK18 analysis, which gives $r_{0.05} < 0.036$ at 95% confidence [7]. Together these results favour plateau-type potentials and disfavour classic large-field monomial models such as $m^2\phi^2$. Future CMB B-mode experiments such as LiteBIRD and CMB-S4 target the $\sigma(r) \sim 10^{-3}$ regime, capable of detecting the tensor signal predicted by Starobinsky inflation. [8, 9]

6. Recent Planck and DESI inflation analyses

The aim of this section is not to memorize every quoted number. Rather, each example shows a different observable that constrains inflation: CMB two-point statistics, the CMB bispectrum, large-scale scale-dependent bias, and the small-scale Lyα power spectrum.

Planck as the CMB anchor. For teaching purposes, the main observational baseline is still the *Planck* 2018 inflation analysis [6], read together if needed with the main cosmological-parameter paper [5]. It found $n_s = 0.9649 \pm 0.0042$ and no evidence for scale dependence of the scalar tilt, while the combination with BK15 tightened the tensor bound to $r_{0.002} < 0.056$ at 95% confidence. The broad lesson is that the data prefer a red tilt, small tensors, and no compelling departure from a nearly power-law adiabatic primordial spectrum.

Planck PR4 and higher-order statistics. A useful recent update comes from the bispectrum analysis of the final *Planck* PR4 temperature and E-polarization maps [14]. That study found

$$f_{\rm NL}^{\rm loc} = -0.1 \pm 5.0, \qquad f_{\rm NL}^{\rm equil} = 6 \pm 46, \qquad f_{\rm NL}^{\rm ortho} = -8 \pm 21, \tag{19}$$

fully consistent with Gaussian initial conditions. For students, this is an important lesson: even after the final *Planck* reprocessing, there is still no statistically significant evidence for the large primordial bispectra predicted by many non-minimal inflationary scenarios.

DESI DR1 local-$f_{\rm NL}$ from LRGs and quasars. The first flagship DESI inflation result came from the large-scale clustering of DR1 LRG and QSO samples [17]. Using 1.63×10^6 LRGs and 1.19×10^6 QSOs, the analysis obtained

$$f_{\rm NL}^{\rm loc} = -3.6^{+9.0}_{-9.1} \qquad (68\%\ \mathrm{CL}), \tag{20}$$

for its baseline modelling of the PNG bias response. This was the most precise large-scale-structure-only local PNG constraint to date, improving over the final eBOSS result by a factor of 2.3. This paper is especially worth showing students because it illustrates how survey blinding, systematics control, and window-function modelling all matter once one pushes to the very largest scales.

DESI and CMB-lensing cross-correlations. Cross-correlations with CMB lensing provide a complementary route because they are less sensitive to some of the large-scale systematics that affect auto-correlations. An earlier DESI-based example used Legacy Survey LRGs cross-correlated with *Planck* PR4 lensing and obtained $f_{\rm NL} = 39^{+40}_{-38}$ from the cross signal alone [15]. A more recent analysis used DESI DR1 quasars and *Planck* PR4 CMB lensing, finding

$$f_{\rm NL} = 2^{+28}_{-34} \quad (p = 1.6), \qquad f_{\rm NL} = 6^{+20}_{-24} \quad (p = 1.0), \tag{21}$$

with roughly a 35% improvement over the earlier DESI quasar-target analysis [18, 16]. The central values remain fully consistent with zero, but the method is attractive because it cleanly illustrates how combining different tracers can test inflation in a robust way.

DESI Lyα 1D power spectrum and running. A different route to inflation uses the small-scale shape of the matter power spectrum. The DESI DR1 Lyα 1D power-spectrum cosmology analysis [19] combined the Lyα measurement with *Planck*, ACT, SPT-3G, and DESI BAO. Relative to the corresponding CMB+BAO combination, it improved the precision on the running α_s and the running-of-running β_s by factors of 1.27 and 1.90, respectively, while remaining fully consistent with a pure power-law primordial spectrum. This is especially instructive for students: DESI adds comparatively little to A_s at the CMB pivot, but it adds substantial leverage on how the spectrum bends across widely separated scales.

The main 2026 lesson. Taken together, the recent *Planck* and DESI analyses paint a consistent picture [6, 14, 17, 18, 19]. The scalar spectrum is red but close to scale invariant, the tensor amplitude is small, the local bispectrum is consistent with zero, and current data do not require running. This does not prove that the Universe underwent the simplest single-field slow-roll inflation, but it keeps that class of models in remarkably good agreement with the data. Conversely, any future detection of $f_{\rm NL}^{\rm loc} \sim \mathcal{O}(1-10)$, significant running, or a sizeable tensor signal would immediately sharpen the model-selection problem.

7. End of Inflation and Reheating

Inflation ends when either ϵ or $|\eta|$ grows to unity. The inflaton then oscillates about the minimum of $V(\phi)$, decaying into standard-model particles and reheating the universe to a temperature

$$T_{\rm reh} \sim \left(\frac{90}{\pi^2 g_*}\right)^{1/4} \sqrt{\Gamma_\phi m_{\rm Pl}}, \tag{22}$$

where Γ_ϕ is the inflaton decay rate.

Reheating and the uncertainty in $N_\star$. When comparing models to data, one should remember that the relevant e-fold number $N_\star$ is not fixed *a priori*. A useful approximate expression is

$$N_\star \simeq 56 - \ln\left(\frac{k_\star}{0.05\,\mathrm{Mpc}^{-1}}\right) + \frac{1}{4}\ln\left(\frac{V_\star^2}{m_{\rm Pl}^4\rho_{\rm end}}\right) + \frac{1-3w_{\rm reh}}{12(1+w_{\rm reh})}\ln\left(\frac{\rho_{\rm reh}}{\rho_{\rm end}}\right), \tag{23}$$

where $w_{\rm reh}$ is an effective reheating equation-of-state parameter. Different reheating histories shift $N_\star$ by several e-folds, which in turn moves a model along its curve in the (n_s, r) plane. This is why modern inflation constraints are often displayed as *bands* rather than single prediction points.

8. Challenges and Alternatives

- **Initial conditions**: Why did the Universe start in the small patch that inflated?
- **Trans-Planckian issues**: modes we observe today originate at sub-Planckian wavelengths.
- **Eternal inflation & multiverse**: stochastic fluctuations can drive perpetual inflation in some regions, raising measure problems.
- **Alternatives**: bouncing or emergent-universe scenarios attempt to solve the same puzzles without inflation.

What would count as a qualitatively new discovery? Examples include a detection of primordial B-modes at $r \gtrsim 10^{-3}$, local primordial non-Gaussianity at observable amplitude, statistically significant running across CMB and Lyα scales, or robust oscillatory features in the primordial spectrum. *Planck* and DESI have not found these signals so far, but they have dramatically narrowed the range in which future discoveries can hide.

9. Summary

Inflation provides a compelling extension of big-bang cosmology that (i) explains the large-scale homogeneity and flatness, (ii) predicts a nearly scale-invariant, Gaussian spectrum of primordial perturbations, and (iii) offers testable tensor-mode signatures. While observations increasingly narrow the viable model space, forthcoming CMB polarization and *21-cm* surveys promise decisive tests in the coming decade.

Homework

(1) For $V(\phi) = m^2\phi^2/2$, use the slow-roll relations to show that $r \simeq 8/N_\star$ and $n_s \simeq 1 - 2/N_\star$. Evaluate these at $N_\star = 50$ and 60, and compare qualitatively with the low-r preference discussed in the lecture.

(2) Starting from

$$\ln \mathcal{P}_\mathcal{R}(k) = \ln A_s + (n_s - 1)\ln(k/k_*) + \frac{\alpha_s}{2}\ln^2(k/k_*) + \cdots,$$

explain in words why adding DESI Lyα data helps more with α_s than with A_s.

(3) Use the scale-dependent bias formula $\Delta b \propto f_{\rm NL}^{\rm loc}/k^2$ to answer the following. If one compares two modes with $k_2 = 2k_1$, by what factor does the PNG signal change? Why are high-bias tracers such as quasars especially valuable?
(4) Current data are consistent with small r, negligible local $f_{\rm NL}$, and no significant running. Name one broad class of inflation models that is favoured by this pattern and one broad class that is disfavoured, and give one sentence of justification for each answer.

Suggested reading for students

For a first pass through the subject, Baumann's *Cosmology* lecture notes (especially Chapters 8–10) and the TASI review [1] are both very accessible. For a book-length treatment, Liddle and Lyth [2] remains a classic. On the observational side, students can begin with the *Planck* inflation paper [6] and then read the recent *Planck* PR4 and DESI examples cited in the main text.

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