LECTURE 15: INFLATION

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1. Motivation for Inflation

The standard (big-bang) cosmology successfully describes the thermal history of the Universe, yet it faces several conceptual challenges:

- **Horizon problem**: the observed CMB is uniform to one part in 10^5 across regions that were causally disconnected at the time of last scattering.
- **Flatness problem**: the present near-critical density ($|\Omega_k| \ll 1$) requires extreme fine-tuning of the initial curvature.
- **Monopole problem**: grand-unified theories generically overproduce magnetic monopoles that are not observed.
- **Origin of perturbations**: the big-bang framework offers no mechanism for the nearly scale-invariant spectrum of primordial fluctuations measured in the CMB.

An early epoch of quasi-exponential expansion—*cosmic inflation*—resolves these problems in a single stroke.

2. Scalar-Field Dynamics

We model inflation with a minimally-coupled scalar field ϕ (the *inflaton*) rolling slowly down its potential $V(\phi)$ in a spatially flat Friedmann–Robertson–Walker (FRW) universe.

$$H^2 = \frac{8\pi G}{3}\Big[\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big], \qquad \dot{H} = -4\pi G\dot{\phi}^2, \tag{1}$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{2}$$

Slow-roll approximation. Inflation occurs if the potential energy dominates, $V \gg \dot{\phi}^2$, which can be recast as the *slow-roll* conditions

$$\epsilon \equiv \frac{m_{\rm Pl}^2}{16\pi}\Big(\frac{V'}{V}\Big)^2 \ll 1, \qquad \eta \equiv \frac{m_{\rm Pl}^2}{8\pi}\frac{V''}{V} \ll 1. \tag{3}$$

Then Eqs. (1)–(2) reduce to

$$H^2 \simeq \frac{8\pi G}{3}V, \qquad 3H\dot{\phi} \simeq -V'. \tag{4}$$

The number of *e-folds* generated between ϕ_i and ϕ_f is

$$N \equiv \int_{t_i}^{t_f} H\mathrm{d}t \simeq \frac{8\pi}{m_{\rm Pl}^2}\int_{\phi_f}^{\phi_i}\frac{V}{V'}\mathrm{d}\phi. \tag{5}$$

Successful inflation requires $N \gtrsim 50$–60 to solve the horizon problem for modes observed today.

3. Prototype Models

A vast zoo of potentials satisfy the slow-roll criteria. Popular examples include

1. **Chaotic inflation**: $V(\phi) = \frac{1}{2}m^2\phi^2$ or $\lambda\phi^4$.
2. **Starobinsky R^2 inflation**: $V(\phi) = \Lambda^4\big(1 - e^{-\sqrt{2/3}\,\phi/m_{\rm Pl}}\big)^2$.
3. **Plateau models** (*e.g.* small-field, hilltop, or E-model potentials).
4. **Hybrid inflation**: inflation ends through a second (waterfall) field.

Each predicts distinct scalar spectral index n_s and tensor-to-scalar ratio r, providing an observational handle to discriminate between them.

4. Generation of Perturbations

Quantum fluctuations of ϕ and the metric are stretched outside the horizon during inflation. In the slow-roll limit, the dimensionless power spectra are

$$\mathcal{P}_{\mathcal{R}}(k) \;=\; A_s\left(\frac{k}{k_*}\right)^{n_s-1},$$

$$n_s - 1 \;\equiv\; \left.\frac{d\ln\mathcal{P}_{\mathcal{R}}(k)}{d\ln k}\right|_{k=k_*}.$$

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2_{k=aH} = \frac{1}{12\pi^2}\frac{V}{m_{\rm Pl}^4\epsilon}, \tag{6}$$

$$\mathcal{P}_h(k) = \frac{2}{3\pi^2}\frac{V}{m_{\rm Pl}^4}, \qquad r \equiv \frac{\mathcal{P}_h}{\mathcal{P}_{\mathcal{R}}} = 16\epsilon. \tag{7}$$

Spectral observables. To first order in slow-roll, the scalar spectral index and its running are

$$n_s - 1 = -6\epsilon + 2\eta, \qquad \frac{{\rm d}n_s}{{\rm d}\ln k} = 16\epsilon\eta - 24\epsilon^2 - 2\xi^2, \tag{8}$$

with $\xi^2 \equiv (m_{\rm Pl}^4/64\pi^2)(V'V'''/V^2)$.

5. Observational Status (*Planck* 2024 + BK18)

Current data constrain $n_s = 0.9649 \pm 0.0042$ and $r < 0.036$ (95% CL), favouring plateau-type potentials while dis-favouring the classic $m^2\phi^2$ model. Future CMB-B-mode experiments (LiteBIRD, CMB-S4) target $\sigma(r) \sim 10^{-3}$, capable of detecting the tensor signal predicted by Starobinsky inflation.

6. End of Inflation and Reheating

Inflation ends when either ϵ or $|\eta|$ grows to unity. The inflaton then oscillates about the minimum of $V(\phi)$, decaying into standard-model particles and reheating the universe to a temperature

$$T_{\rm reh} \sim \left(\frac{90}{\pi^2 g_*}\right)^{1/4} \sqrt{\Gamma_\phi m_{\rm Pl}}, \tag{9}$$

where Γ_ϕ is the inflaton decay rate.

7. Challenges and Alternatives

- **Initial conditions**: Why did the Universe start in the small patch that inflated?
- **Trans-Planckian issues**: modes we observe today originate at sub-Planckian wavelengths.
- **Eternal inflation & multiverse**: stochastic fluctuations can drive perpetual inflation in some regions, raising measure problems.
- **Alternatives**: bouncing or emergent-universe scenarios attempt to solve the same puzzles without inflation.

8. Summary

Inflation provides a compelling extension of big-bang cosmology that (i) explains the large-scale homogeneity and flatness, (ii) predicts a nearly scale-invariant, Gaussian spectrum of primordial perturbations, and (iii) offers testable tensor-mode signatures. While observations increasingly narrow the viable model space, forthcoming CMB polarization and *21-cm* surveys promise decisive tests in the coming decade.

Further reading

(1) D. Baumann, *Cosmology* (Lecture Notes), Chapters 8–10.
(2) A. R. Liddle & D. H. Lyth, *Cosmological Inflation and Large-Scale Structure* (CUP, 2000).
(3) L. Peiris *et al.*, "First-Year *Wilkinson Microwave Anisotropy Probe* (WMAP) Observations: Implications for Inflation," *Astrophys. J. Suppl.* **148**, 213 (2003), arXiv:astro-ph/0302225.
(4) G. Hinshaw *et al.*, "Nine-Year *Wilkinson Microwave Anisotropy Probe* (WMAP) Observations: Cosmological Parameter Results," *Astrophys. J. Suppl.* **208**, 19 (2013), arXiv:1212.5226.
(5) Planck Collaboration X, "Planck 2018 Results. X. Constraints on Inflation," *Astron. Astrophys.* extbf641, A10 (2020), arXiv:1807.06211.
(6) Planck Collaboration, "Planck 2018 results. VI. Cosmological parameters (*A&A* **641**, A6, 2020; arXiv:1807.06209)."
(7) DESI Collaboration, "DESI DR2: Constraints on Inflation from Large-Scale Structure," *arXiv*:2505.17890 (2025).