Solutions to the Homework in Lecture 12
Fundamentals of Observational Cosmology

These notes provide worked solutions to the homework problems in Lecture 12. For Problem 2, a standalone Julia script is also included in the file `lecture12_weighted_paircounts.jl`.

Problem 1. Effective volume

Question. Starting from the FKP weight, derive $V_{\rm eff}(k)$ and explain the limiting behaviour when $\bar{n}P \gg 1$ and when $\bar{n}P \ll 1$.

Solution. In the lecture, the shell-averaged power-spectrum variance is written schematically as

$$\frac{\sigma_P^2(k)}{P^2(k)} = \frac{(2\pi)^3}{V_k} \frac{\int \mathrm{d}^3r\, \bar{n}^4(\mathbf{r}) w^4(\mathbf{r}) \left[1 + \frac{1}{\bar{n}(\mathbf{r})P(k)}\right]^2}{\left[\int \mathrm{d}^3r\, \bar{n}^2(\mathbf{r}) w^2(\mathbf{r})\right]^2}.$$

The FKP weight that minimizes this expression is

$$w_{\rm FKP}(\mathbf{r}) = \frac{1}{1 + \bar{n}(\mathbf{r})P(k)}.$$

Insert this weight into the variance formula. In the numerator,

$$\begin{aligned} \bar{n}^4 w_{\rm FKP}^4 \left(1 + \frac{1}{\bar{n}P}\right)^2 &= \frac{\bar{n}^4}{(1+\bar{n}P)^4} \frac{(1+\bar{n}P)^2}{(\bar{n}P)^2} \\ &= \frac{\bar{n}^2}{(1+\bar{n}P)^2} \frac{1}{P^2}. \end{aligned}$$

The denominator contains

$$\bar{n}^2 w_{\rm FKP}^2 = \frac{\bar{n}^2}{(1+\bar{n}P)^2}.$$

Therefore

$$\frac{\sigma_P^2(k)}{P^2(k)} \propto \frac{(2\pi)^3}{V_k} \left[\int \mathrm{d}^3r \, \frac{\bar{n}^2(\mathbf{r})P^2(k)}{[1+\bar{n}(\mathbf{r})P(k)]^2}\right]^{-1}.$$

This motivates the definition

$$\boxed{V_{\rm eff}(k) = \int \mathrm{d}^3r \left[\frac{\bar{n}(\mathbf{r})P(k)}{1+\bar{n}(\mathbf{r})P(k)}\right]^2.}$$

With this definition, the usual Gaussian result is written as

$$\mathrm{Var}\big[\hat{P}(k)\big] \simeq \frac{2(2\pi)^3}{V_k V_{\rm eff}(k)} P^2(k),$$

or equivalently

$$N_{\rm mode}(k) \simeq \frac{V_k V_{\rm eff}(k)}{(2\pi)^3}, \qquad \frac{\sigma_P(k)}{P(k)} \simeq \sqrt{\frac{2}{N_{\rm mode}(k)}}.$$

So $V_{\rm eff}$ is the volume of an ideal uniform survey that would give the same statistical power as the real inhomogeneous survey.

Limit 1: $\bar{n}P \gg 1$. In a dense region,

$$\frac{\bar{n}P}{1+\bar{n}P} \to 1,$$

so the integrand tends to unity. Hence

$$V_{\rm eff}(k) \to \int d^3r = V_{\rm survey}.$$

This is the cosmic-variance-dominated regime. Once the galaxy density is high enough, adding more galaxies hardly helps; the survey is already using nearly all of its geometric volume.

Limit 2: $\bar{n}P \ll 1$. In a sparse region,

$$\frac{\bar{n}P}{1+\bar{n}P} \approx \bar{n}P,$$

so

$$V_{\rm eff}(k) \approx \int d^3r\, [\bar{n}(\mathbf{r})P(k)]^2.$$

If $\bar{n}$ is roughly constant over a region of volume V, this becomes

$$V_{\rm eff}(k) \approx (\bar{n}P)^2 V \ll V.$$

This is the shot-noise-dominated regime. Such regions contribute little to the measurement because there are too few galaxies to beat the Poisson noise.

So the physical meaning is very clear: the factor

$$\left[\frac{\bar{n}P}{1+\bar{n}P}\right]^2$$

measures how useful each volume element is for estimating $P(k)$.

Problem 2. Anisotropic pair counts

Question. Write pseudo-code for measuring $DD(s,\mu)$, $DR(s,\mu)$ and $RR(s,\mu)$ using weighted pair counts.

Solution. For weighted anisotropic pair counts, the definitions are

$$DD(s,\mu) = \frac{\sum_{i<j\in D} w_i w_j\, \Theta_{ij}(s,\mu)}{\sum_{i<j\in D} w_i w_j}, \qquad DR(s,\mu) = \frac{\sum_{i\in D}\sum_{j\in R} w_i w_j\, \Theta_{ij}(s,\mu)}{\sum_{i\in D}\sum_{j\in R} w_i w_j},$$

$$RR(s,\mu) = \frac{\sum_{i<j\in R} w_i w_j\, \Theta_{ij}(s,\mu)}{\sum_{i<j\in R} w_i w_j},$$

where $\Theta_{ij}(s,\mu) = 1$ if the pair lies in the chosen (s,μ) bin and 0 otherwise.

A direct Julia-style pseudocode implementation is:

```
# Inputs:
#   data_xyz   :: Nd x 3 array of Cartesian positions
#   rand_xyz   :: Nr x 3 array of Cartesian positions
#   w_data     :: length-Nd vector of weights
#   w_rand     :: length-Nr vector of weights
#   s_edges    :: bin edges in separation
#   mu_edges   :: bin edges in mu (usually from 0 to 1)
```

```
initialize DD_hist[ns, nmu] = 0
initialize DR_hist[ns, nmu] = 0
initialize RR_hist[ns, nmu] = 0

norm_DD = 0.5 * ((sum(w_data))^2 - sum(w_data.^2))
norm_RR = 0.5 * ((sum(w_rand))^2 - sum(w_rand.^2))
norm_DR = sum(w_data) * sum(w_rand)

# DD pairs
for i in 1:(Nd-1)
    for j in (i+1):Nd
        s_vec = data_xyz[j, :] - data_xyz[i, :]
        s     = norm(s_vec)
        n_vec = data_xyz[j, :] + data_xyz[i, :]
        mu    = abs(dot(s_vec, n_vec) / (norm(s_vec) * norm(n_vec)))
        (is, im) = find_bin_indices(s, mu, s_edges, mu_edges)
        if (is, im) is a valid bin
            DD_hist[is, im] += w_data[i] * w_data[j]
        end
    end
end

# DR pairs
for i in 1:Nd
    for j in 1:Nr
        s_vec = rand_xyz[j, :] - data_xyz[i, :]
        s     = norm(s_vec)
        n_vec = rand_xyz[j, :] + data_xyz[i, :]
        mu    = abs(dot(s_vec, n_vec) / (norm(s_vec) * norm(n_vec)))
        (is, im) = find_bin_indices(s, mu, s_edges, mu_edges)
        if (is, im) is a valid bin
            DR_hist[is, im] += w_data[i] * w_rand[j]
        end
    end
end

# RR pairs
for i in 1:(Nr-1)
    for j in (i+1):Nr
        s_vec = rand_xyz[j, :] - rand_xyz[i, :]
        s     = norm(s_vec)
        n_vec = rand_xyz[j, :] + rand_xyz[i, :]
        mu    = abs(dot(s_vec, n_vec) / (norm(s_vec) * norm(n_vec)))
        (is, im) = find_bin_indices(s, mu, s_edges, mu_edges)
        if (is, im) is a valid bin
            RR_hist[is, im] += w_rand[i] * w_rand[j]
        end
    end
end

DD = DD_hist / norm_DD
DR = DR_hist / norm_DR
RR = RR_hist / norm_RR

xi = (DD - 2 .* DR + RR) ./ RR     # anisotropic Landy-Szalay estimator
```

Some comments help interpret this algorithm:

- For each pair, the separation is $\mathbf{s} = \mathbf{x}_2 - \mathbf{x}_1$, the midpoint line of sight is $\hat{\mathbf{n}} = (\mathbf{x}_1 + \mathbf{x}_2)/|\mathbf{x}_1 + \mathbf{x}_2|$, and $\mu = \hat{\mathbf{s}} \cdot \hat{\mathbf{n}}$.
- The pair weight is the product $w_i w_j$. In practice each object weight may already include systematic, close-pair, redshift-failure, FKP, and possibly redshift weights.

- The normalizations make the histograms dimensionless and match the usual Landy-Szalay conventions.
- For a real survey one does *not* keep the brute-force loops; instead one uses a kd-tree or another accelerated search. The counting logic, however, is exactly the same.

I also provide a standalone Julia file, `lecture12_weighted_paircounts.jl`, that implements the same logic in a compact pedagogical script.

Problem 3. Effective redshift

Question. Explain why a sign-changing redshift weight should be compared to a redshift-weighted model rather than to a model evaluated at one effective redshift.

Solution. For an ordinary unweighted measurement, it is often reasonable to summarize the sample by a single pair-weighted effective redshift. But a redshift-weighted measurement is not, in general, the same thing as an unweighted measurement performed at one special redshift.

The reason is that the measured statistic is a *weighted integral* over the full redshift range:

$$M_w = \frac{1}{N} \int \mathrm{d}\mathcal{W}(z)\, w(z) M(z),$$

where $M(z)$ is the theoretical model and $\mathrm{d}\mathcal{W}(z)$ includes the pair-weighting and inverse-variance factors. The correct theory prediction is therefore the *same weighted integral* applied to the model.

If one tries to define a single effective redshift by

$$z_{\mathrm{eff}} = \frac{\int \mathrm{d}\mathcal{W}(z)\, w(z) z}{\int \mathrm{d}\mathcal{W}(z)\, w(z)},$$

problems appear as soon as $w(z)$ changes sign:

- the denominator can become very small, so z_{eff} becomes unstable or even undefined;
- positive and negative parts of the weight cancel, so the resulting number does not tell you where the information really comes from;
- even if z_{eff} is finite, there is generally no single redshift z_* for which $M_w = M(z_*)$.

The last point is easiest to see by Taylor expanding the model around a pivot redshift z_p:

$$M(z) = M(z_p) + M'(z_p)(z - z_p) + \frac{1}{2} M''(z_p)(z - z_p)^2 + \cdots .$$

Then the weighted measurement becomes

$$M_w = A_0 M(z_p) + A_1 M'(z_p) + \frac{1}{2} A_2 M''(z_p) + \cdots ,$$

with moments

$$A_n = \frac{1}{N} \int \mathrm{d}\mathcal{W}(z)\, w(z)(z - z_p)^n.$$

For sign-changing weights, the zeroth moment A_0 can be small or vanish, while A_1 or A_2 is deliberately made important. In other words, such weights are designed to measure *gradient* or *curvature* information across redshift, not just an average value at one redshift.

Therefore the correct comparison is

$$\boxed{M_w = \frac{1}{N}\int \mathrm{d}\mathcal{W}(z)\, w(z) M(z),}$$

not

$$M(z_{\rm eff}).$$

A single effective redshift is at best a rough mnemonic; it is not the correct theory prediction when the redshift weight changes sign.