Solutions to the Homework in Lecture 11
Fundamentals of Observational Cosmology

These notes provide worked solutions to the homework problems in Lecture 11. Throughout, we use the multipole convention

$$P_\ell(k) = \frac{2\ell+1}{2}\int_{-1}^{1} \mathrm{d}\mu\, P(k,\mu)\mathcal{L}_\ell(\mu),$$

and the useful Legendre decompositions

$$\mu^0 = \mathcal{L}_0(\mu), \qquad \mu^2 = \frac{1}{3}\mathcal{L}_0(\mu) + \frac{2}{3}\mathcal{L}_2(\mu), \qquad \mu^4 = \frac{1}{5}\mathcal{L}_0(\mu) + \frac{4}{7}\mathcal{L}_2(\mu) + \frac{8}{35}\mathcal{L}_4(\mu).$$

Problem 1. Kaiser multipoles

Question. Starting from

$$P^s(k,\mu) = (b + f\mu^2)^2 P_\mathrm{m}(k),$$

derive $P_0(k)$, $P_2(k)$, and $P_4(k)$.

Solution. Expand the square first:

$$P^s(k,\mu) = \left[b^2 + 2bf\mu^2 + f^2\mu^4\right] P_\mathrm{m}(k).$$

Now insert the Legendre decompositions of μ^2 and μ^4:

$$\begin{aligned} P^s(k,\mu) &= \left[b^2 + 2bf\left(\frac{1}{3}\mathcal{L}_0 + \frac{2}{3}\mathcal{L}_2\right) + f^2\left(\frac{1}{5}\mathcal{L}_0 + \frac{4}{7}\mathcal{L}_2 + \frac{8}{35}\mathcal{L}_4\right)\right] P_\mathrm{m}(k) \\ &= \left[\left(b^2 + \frac{2}{3}bf + \frac{1}{5}f^2\right)\mathcal{L}_0 + \left(\frac{4}{3}bf + \frac{4}{7}f^2\right)\mathcal{L}_2 + \frac{8}{35}f^2\mathcal{L}_4\right] P_\mathrm{m}(k). \end{aligned}$$

Therefore the non-zero even multipoles are

$$\boxed{P_0(k) = \left(b^2 + \frac{2}{3}bf + \frac{1}{5}f^2\right) P_\mathrm{m}(k),}$$

$$\boxed{P_2(k) = \left(\frac{4}{3}bf + \frac{4}{7}f^2\right) P_\mathrm{m}(k),}$$

$$\boxed{P_4(k) = \frac{8}{35}f^2 P_\mathrm{m}(k).}$$

All odd multipoles vanish in the distant-observer, parity-symmetric limit because the model contains only even powers of μ.

It is also common to define $\beta \equiv f/b$. Then

$$P_0(k) = b^2\left(1 + \frac{2}{3}\beta + \frac{1}{5}\beta^2\right) P_\mathrm{m}(k), \qquad P_2(k) = b^2\left(\frac{4}{3}\beta + \frac{4}{7}\beta^2\right) P_\mathrm{m}(k), \qquad P_4(k) = b^2\frac{8}{35}\beta^2 P_\mathrm{m}(k).$$

Problem 2. Cross-spectrum multipoles

Question. Starting from

$$P^{AB}(k,\mu) = (b_A + f\mu^2)(b_B + f\mu^2)P_{\rm m}(k),$$

derive $P_0^{AB}(k)$, $P_2^{AB}(k)$, and $P_4^{AB}(k)$.

Solution. First expand the product:

$$P^{AB}(k,\mu) = \left[b_A b_B + f(b_A+b_B)\mu^2 + f^2\mu^4\right] P_{\rm m}(k).$$

Using the same Legendre identities as above,

$$P^{AB}(k,\mu) = \Bigg[b_A b_B + f(b_A+b_B)\left(\frac{1}{3}\mathcal{L}_0 + \frac{2}{3}\mathcal{L}_2\right) + f^2\left(\frac{1}{5}\mathcal{L}_0 + \frac{4}{7}\mathcal{L}_2 + \frac{8}{35}\mathcal{L}_4\right)\Bigg] P_{\rm m}(k).$$

Collecting the coefficients of each Legendre polynomial gives

$$\boxed{P_0^{AB}(k) = \left[b_A b_B + \frac{1}{3}f(b_A+b_B) + \frac{1}{5}f^2\right] P_{\rm m}(k),}$$

$$\boxed{P_2^{AB}(k) = \left[\frac{2}{3}f(b_A+b_B) + \frac{4}{7}f^2\right] P_{\rm m}(k),}$$

$$\boxed{P_4^{AB}(k) = \frac{8}{35}f^2 P_{\rm m}(k).}$$

As a useful check, if we set $A = B$ so that $b_A = b_B = b$, these expressions reduce exactly to the auto-spectrum Kaiser multipoles from Problem 1.

Problem 3. Configuration-space connection

Question. Use the spherical-Bessel relation between $P_\ell(k)$ and $\xi_\ell(s)$ to show why the same physical anisotropy can be discussed in Fourier space or configuration space.

Solution. The anisotropic two-point statistics in Fourier space and configuration space are Fourier transforms of the same underlying quantity:

$$\xi(\mathbf{s}) = \int \frac{{\rm d}^3k}{(2\pi)^3} P(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{s}}.$$

In the distant-observer limit we expand both statistics in Legendre multipoles,

$$P(\mathbf{k}) = \sum_\ell P_\ell(k)\mathcal{L}_\ell(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}), \qquad \xi(\mathbf{s}) = \sum_\ell \xi_\ell(s)\mathcal{L}_\ell(\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}),$$

where $\hat{\mathbf{n}}$ is the line-of-sight direction.

The plane wave can be expanded as

$$e^{i\mathbf{k}\cdot\mathbf{s}} = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell j_\ell(ks)\mathcal{L}_\ell(\hat{\mathbf{k}}\cdot\hat{\mathbf{s}}).$$

Using this expansion and the orthogonality of Legendre polynomials, one finds that each multipole transforms independently:

$$\boxed{\xi_\ell(s) = i^\ell \int \frac{k^2 \mathrm{d}k}{2\pi^2}\, P_\ell(k) j_\ell(ks),}$$

and, conversely,

$$\boxed{P_\ell(k) = 4\pi(-i)^\ell \int s^2 \mathrm{d}s\, \xi_\ell(s) j_\ell(ks).}$$

These are spherical-Bessel transforms.

The important conceptual point is that the transform is one-to-one and acts multipole by multipole. Therefore:

- if redshift-space distortions generate a non-zero quadrupole $P_2(k)$ in Fourier space, then there is a corresponding non-zero quadrupole $\xi_2(s)$ in configuration space;
- if the anisotropy also produces a hexadecapole $P_4(k)$, it maps to a hexadecapole $\xi_4(s)$;
- the physics is unchanged - only the representation is different.

For example, the linear Kaiser effect produces the multipoles from Problem 1 in $P_\ell(k)$. The same effect appears in configuration space after the spherical-Bessel transform, so one may analyze the anisotropy either as oscillatory features in $P_\ell(k)$ or as anisotropic clustering in $\xi_\ell(s)$.

In short, Fourier space and configuration space are two equivalent descriptions of the same anisotropic two-point information. One representation may be more convenient for theory or measurement, but the underlying physical signal is the same.

Problem 4. BAO reconstruction

Question. Explain why the reconstructed field is written as $\delta_{\text{rec}} = \delta_d - \delta_s$, and describe the qualitative difference between the *Rec-Iso* and *Rec-Sym* conventions.

Solution. BAO reconstruction tries to undo the large-scale bulk flows that smear out the acoustic peak. The basic idea is:

1. estimate a large-scale displacement field from a smoothed density field;
2. move the galaxies backward by this estimated displacement to form the *displaced* field δ_d;
3. move an unclustered random catalogue through the same survey mask to form the *shifted* field δ_s;
4. subtract the shifted field from the displaced field.

This gives

$$\delta_{\text{rec}} = \delta_d - \delta_s.$$

Why is the subtraction necessary? Because δ_d by itself still contains the survey selection function and mean-density contribution. The shifted random field δ_s is a smooth reference field with the *same* mask and radial selection but no intrinsic clustering. Subtracting it removes the smooth background and leaves a reconstructed overdensity field. In compact catalogue notation this is exactly analogous to the usual pre-reconstruction field,

$$F^{\text{rec}}(\mathbf{r}) = w(\mathbf{r}) \left[n_d^{\text{disp}}(\mathbf{r}) - \alpha n_s^{\text{shift}}(\mathbf{r}) \right].$$

So the reconstructed field keeps the clustering signal while canceling the survey geometry and mean-density terms.

The qualitative difference between the two common conventions is the treatment of large-scale redshift-space distortions:

- **Rec-Iso (isotropic reconstruction).** The procedure is designed to remove most of the large-scale Kaiser anisotropy from the reconstructed field. As a result, the post-reconstruction field is closer to isotropic and the post-reconstruction quadrupole is usually small. This is often preferred for BAO-only analyses.
- **Rec-Sym (symmetric reconstruction).** The procedure keeps the large-scale redshift-space distortion signal in a more symmetric way between galaxies and randoms. The post-reconstruction field therefore still has a non-negligible quadrupole, so both monopole and quadrupole can be modeled and fitted.

In both cases the main purpose is the same: sharpen the BAO feature by partially reversing large-scale bulk motions. The difference is whether one tries to remove the large-scale RSD anisotropy at the same time.