LECTURE 8: FUNDAMENTALS OF OBSERVATIONAL COSMOLOGY (IV)

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1. Performing a Fisher matrix analysis for SN Ia

In this lecture, we plan to perform a Fisher matrix analysis using the data covariance of the SN Ia data, and compare the result with that obtained from a full χ^2 analysis.

Recall that for SN Ia, the Fisher matrix is,

$$\mathbf{F}_{ij} = \left(\frac{\partial\mu}{\partial\theta_i}\right)^T \mathbf{\Sigma}^{-1}\frac{\partial\mu}{\partial\theta_j} \tag{1}$$

So what is needed is to evaluate the derivatives $\frac{\partial\mu}{\partial\theta_i}$ numerically. In practice, we use finite difference (FD) to approximate derivatives, *i.e.*,

$$\frac{\mathrm{d}f}{\mathrm{d}x} \simeq \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x} \tag{2}$$

So

$$\frac{\partial\mu}{\partial\Omega_{\mathrm{M}}} \simeq \frac{\mu\left(\Omega_{\mathrm{M}}+\Delta\Omega_{\mathrm{M}},\Omega_\Lambda\right)-\mu\left(\Omega_{\mathrm{M}}-\Delta\Omega_{\mathrm{M}},\Omega_\Lambda\right)}{2\Delta\Omega_{\mathrm{M}}} \tag{3}$$

$$\frac{\partial\mu}{\partial\Omega_\Lambda} \simeq \frac{\mu\left(\Omega_{\mathrm{M}},\Omega_\Lambda+\Delta\Omega_\Lambda\right)-\mu\left(\Omega_{\mathrm{M}},\Omega_\Lambda-\Delta\Omega_\Lambda\right)}{2\Delta\Omega_\Lambda} \tag{4}$$

2. Contour plotting from a covariance matrix

Let $\mathbf{C}$ be the covariance matrix for parameters x and y after marginalising over other parameters. Denote

$$\mathbf{C} \equiv \begin{pmatrix} \sigma_{\mathrm{x}}^2 & \rho\sigma_{\mathrm{x}}\sigma_{\mathrm{y}} \\ \rho\sigma_{\mathrm{x}}\sigma_{\mathrm{y}} & \sigma_{\mathrm{y}}^2 \end{pmatrix} \tag{5}$$

The χ^2 for the bivariate $\boldsymbol{P} \equiv (\mathrm{x},\mathrm{y})^T$ is simply

$$\Delta\chi^2 \equiv \boldsymbol{P}^T\mathbf{C}^{-1}\boldsymbol{P} = \frac{1}{1-\rho^2}\left(\frac{\mathrm{x}^2}{\sigma_{\mathrm{x}}^2} - 2\frac{\rho\mathrm{xy}}{\sigma_{\mathrm{x}}\sigma_{\mathrm{y}}} + \frac{\mathrm{y}^2}{\sigma_{\mathrm{y}}^2}\right) \tag{6}$$

where I have assumed that the mean values for x and y are subtracted off. Given $\mathbf{C}$ one can draw the error contour by setting $\Delta\chi^2$ to a constant depending on the confidence level required. This is mathematically transparent but not practically efficient, namely, one has to sample x and y on a sufficiently fine grid, calculate $\Delta\chi^2$ and then minimise it. This is tedious when we need to draw multiple contours at the same time, which is usually the case when performing future forecasts.

The contour plots can be made much faster by plotting the error ellipses directly using the parametric equations for the ellipses. In Eq (6), it is obvious that the error contour is in general a tilted ellipse when $\Delta\chi^2$ is set to be a constant depending on the confidence level required. The parametric equation for a standard untilted ellipse is

$$\begin{aligned} \mathrm{x}'(t) &= a\ \cos\ t \\ \mathrm{y}'(t) &= b\ \sin\ t \end{aligned} \tag{7}$$

where a and b are the major and minor axes respectively and $t \in [0,\ 2\pi]$. For the general case, e.g., the tilted ellipse with a counterclockwise rotation angle θ (see Fig 2), the parametric equation can be obtained using the rotation matrix R, i.e.,

$$V = RV',\ V \equiv \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix},\ V' \equiv \begin{pmatrix} \mathrm{x}' \\ \mathrm{y}' \end{pmatrix},\ R \equiv \begin{pmatrix} \cos\ \theta & -\sin\ \theta \\ \sin\ \theta & \cos\ \theta \end{pmatrix} \tag{8}$$

where x and y are the coordinates of the tilted ellipse. The parametric equation for the tilted ellipse reads,

$$\begin{aligned} \mathrm{x}(t) &= (a\ \cos\ \theta)\ \cos\ t - (b\ \sin\ \theta)\ \sin\ t \\ \mathrm{y}(t) &= (a\ \sin\ \theta)\ \cos\ t + (b\ \cos\ \theta)\ \sin\ t \end{aligned} \tag{9}$$

So what we need is to find a, b and θ. Note that

$$\mathbf{C} \equiv \left\langle VV^T \right\rangle - \langle V \rangle \left\langle V^T \right\rangle = R\ \mathbf{C}'\ R^T \tag{10}$$

where

$$\mathbf{C}' \equiv \left\langle V'V'^T \right\rangle - \left\langle V' \right\rangle \left\langle V'^T \right\rangle = R^T\ \mathbf{C}\ R \tag{11}$$

In the last step we have used Eq (10) and the fact that $R^T R = I$. Since x' and y' are the coordinates for the untilted ellipse, $\mathbf{C}'$ is diagonal. Combining Eqs (5), (6) and (8), we get,

$$\begin{aligned} a^2 &= \Delta\chi^2\ \mathbf{C}'(1,1) = \Delta\chi^2 \left(\sigma_\mathrm{x}^2\ \cos^2\theta + \rho\sigma_\mathrm{x}\sigma_\mathrm{y}\ \sin\ 2\theta + \sigma_\mathrm{y}^2\ \sin^2\theta\right) \\ b^2 &= \Delta\chi^2\ \mathbf{C}'(2,2) = \Delta\chi^2 \left(\sigma_\mathrm{x}^2\ \sin^2\theta - \rho\sigma_\mathrm{x}\sigma_\mathrm{y}\ \sin\ 2\theta + \sigma_\mathrm{y}^2\ \cos^2\theta\right) \\ 0 &= \mathbf{C}'(1,2) = \frac{\cos\ 2\theta}{2}\left[\left(\sigma_\mathrm{x}^2 - \sigma_\mathrm{y}^2\right)\tan\ 2\theta + 2\rho\sigma_\mathrm{x}\sigma_\mathrm{y}\right] \end{aligned} \tag{12}$$

Therefore

$$\tan\ 2\theta = \frac{2\rho\sigma_\mathrm{x}\sigma_\mathrm{y}}{\sigma_\mathrm{x}^2 - \sigma_\mathrm{y}^2} \tag{13}$$

Plugging Eqs (12) and (13) into Eq (9), we can obtain the parametric equation for the error ellipse given the covariance matrix. Note that for two-dimensional joint likelihood distributions, $\Delta\chi^2 \simeq 2.31,\ 6.17$ and 11.8 for $1,\ 2$ and 3σ ($68.3,\ 95.4$ and 99.7% C.L.) respectively.

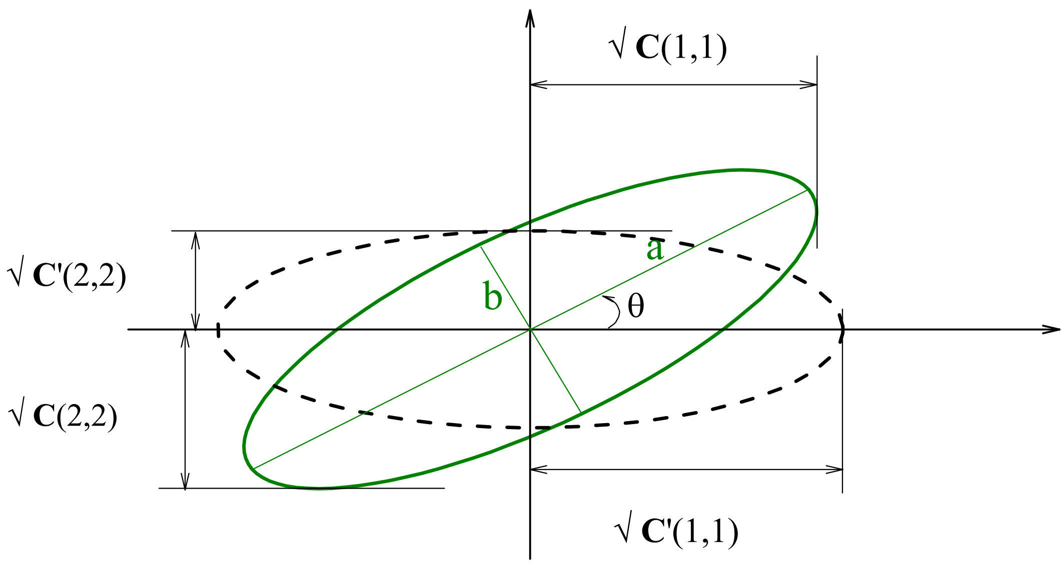

FIGURE 1. The illustration of the error ellipse. The green solid ellipse represents a general error contour with a tilt, and the black dashed one shows the rotated ellipse so that it is untilted. The covariance matrices for the titled and untilted ellipses are $\mathbf{C}$ and $\mathbf{C}'$ respectively and $\mathbf{C}'$ is diagonal.

References

[1] M. Tegmark, A. Taylor and A. Heavens, Astrophys. J. **480**, 22 (1997) doi:10.1086/303939 [astro-ph/9603021].

[2] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, "Numerical Recipes in FORTRAN: The Art of Scientific Computing."